

Discrete Modeling

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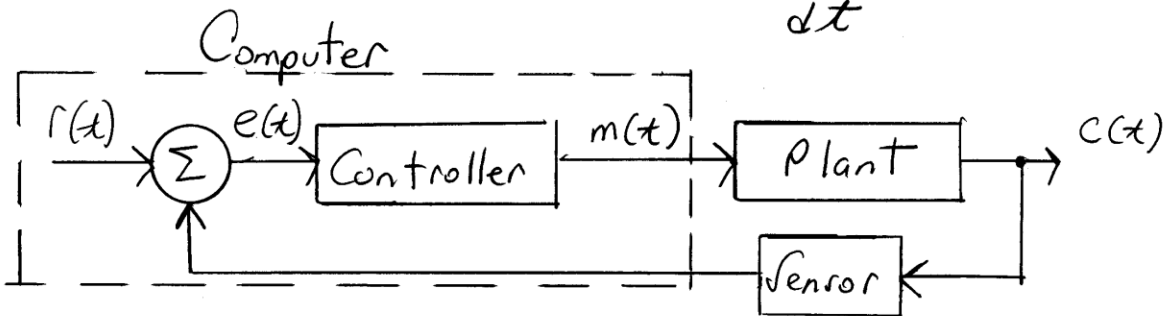
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Plant: 1st Order

$$\tau \frac{dc(t)}{dt} + c(t) = K m(t)$$

Controller: PID

$$m(t) = K_p e(t) + K_I \int_0^t e(\tau) d\tau + K_D \frac{de(t)}{dt}$$



Computer-Controlled Closed-Loop System

- Assume:
- 1) Process dynamics are known
 - 2) Sensor is ideal
 - 3) Computer output is a staircase function (sample-and-hold)
 - 4) Time required to sample process output and produce process input $\ll T$ (sample period)

Consider the 1st-Order process:

$$\tau \frac{dc(t)}{dt} + c(t) = Km(t)$$

$$\left\{ \begin{array}{l} \tau = \text{time constant} \\ K = \text{process gain} \end{array} \right.$$

At $t=0$, $c(0) = C_0$ I.C.

Input: $m(t) = m_0 = \text{constant}$ $0 \leq t < T$

Solution: $c(t) = C e^{-t/\tau} + Km_0$

From I.C.: $C = C_0 - Km_0$

$$\text{Thus: } c(t) = C_0 e^{-t/\tau} + Km_0(1 - e^{-t/\tau})$$

$$c(T) = C_0 e^{-T/\tau} + Km_0(1 - e^{-T/\tau})$$

Corresponding 1st-Order Difference Equation:

$$\left. \begin{array}{l} C_n - C_{n-1} e^{-T/\tau} = Km_{n-1}(1 - e^{-T/\tau}) \\ C_n = C_{n-1}(e^{-T/\tau}) + m_{n-1}K(1 - e^{-T/\tau}) \end{array} \right\} \begin{array}{l} \text{Recursion} \\ \text{Method} \end{array}$$

This yields exact response values at sample instants assuming the input is held constant during any sample period.

Consider the Controller :

- obtain sample process value C_n
- calculate error $e_n = r_n - C_n$
(r_n = desired value or command)
- computer manipulated process input m_n
- outputs m_n to plant

Assume: input to the process is a sequence of constant values that change instantaneously at the beginning of each sample period.

Typical control algorithm :

$$m_n = m_{n-1} + K_0 e_n + K_1 e_{n-1} + K_2 e_{n-2} + \dots$$

Consider a P/D Controller

(a) Proportional Control

$$m(t) = K_p e(t)$$

$$\left. \begin{array}{l} m_n = K_p e_n \\ m_{n-1} = K_p e_{n-1} \end{array} \right\} m_n - m_{n-1} = K_p (e_n - e_{n-1})$$

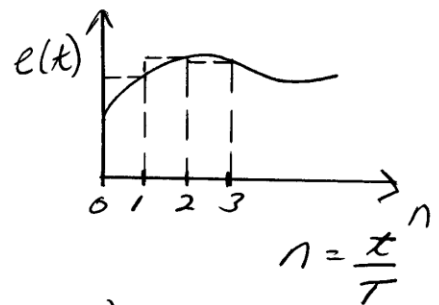
(b) Integral Control

$$m(t) = K_I \int_0^t e(\tau) d\tau$$

$$m_n = K_I \sum_{j=1}^n T e_j$$

backward
rectangle
rule

$$\begin{cases} m_n = K_I \sum_{j=1}^{n-1} T e_j + K_I T e_n \\ m_n = m_{n-1} + K_I T e_n \end{cases}$$

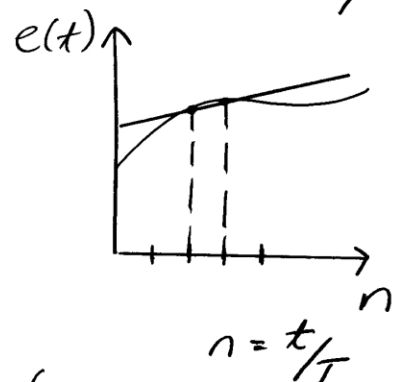


$$m_n - m_{n-1} = K_I T e_n$$

(c) Derivative Control

$$m(t) = K_D \frac{de(t)}{dt}$$

$$\frac{de(t)}{dt} = \frac{e_n - e_{n-1}}{T} \quad \left\{ \begin{array}{l} \text{backward} \\ \text{difference} \\ \text{rule} \end{array} \right.$$



$$\begin{cases} m_n = K_D \left(\frac{e_n - e_{n-1}}{T} \right) \\ m_{n-1} = K_D \left(\frac{e_{n-1} - e_{n-2}}{T} \right) \end{cases} \quad \left\{ \begin{array}{l} m_n - m_{n-1} = \\ \frac{K_D}{T} (e_n - 2e_{n-1} + e_{n-2}) \end{array} \right.$$

(d) Summary: PID Control

$$\begin{aligned}
 m_n - m_{n-1} &= K_0 e_n + K_1 e_{n-1} + K_2 e_{n-2} \\
 &= K_p (e_n - e_{n-1}) \\
 &\quad + K_I T e_n \\
 &\quad + \frac{K_0}{T} (e_n - 2e_{n-1} + e_{n-2})
 \end{aligned}$$

$$\left\{ \begin{array}{l} K_0 = K_p + K_I T + \frac{K_0}{T} \\ K_1 = -K_p - \frac{2K_0}{T} \\ K_2 = \frac{K_0}{T} \end{array} \right\} \Rightarrow \text{Functions of } T$$

Note: differences between magnitudes of the coefficients are often just as important as their relative magnitudes.

Back-Shift Operator B :

- obtain discrete TFR
- develop block diagrams
- use algebraic techniques for manipulation purposes

$$\left. \begin{aligned} B y_n &= y_{n-1} \\ B^2 y_n &= y_{n-2} \\ B^j y_n &= y_{n-j} \end{aligned} \right\} \text{back-shift operator}$$

Process : $\tau \frac{dc(t)}{dt} + c(t) = K m(t)$

$$c_n = c_{n-1} (e^{-T/\tau}) + m_{n-1} K (1 - e^{-T/\tau})$$

$$c_n = B c_n (e^{-T/\tau}) + B m_n K (1 - e^{-T/\tau})$$

$$c_n (1 - B e^{-T/\tau}) = m_n B K (1 - e^{-T/\tau})$$

$$\frac{c_n}{m_n} = \frac{K (1 - e^{-T/\tau}) B}{1 - e^{-T/\tau} B} = G_p(B)$$

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Controller: $m_n - m_{n-1} = K_0 e_n + K_1 e_{n-1} + K_2 e_{n-2}$

for a PI Controller: $K_0 = 0$

$$\begin{cases} K_0 = K_p + K_I T \\ K_1 = -K_p \\ K_2 = 0 \end{cases}$$

$$m_n - m_{n-1} = (K_p + K_I T) e_n - K_p e_{n-1}$$

$$m_n - \beta m_n = (K_p + K_I T) e_n - K_p \beta e_n$$

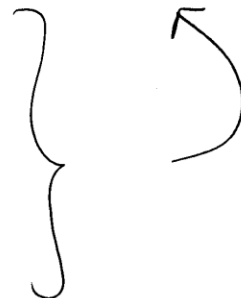
$$\frac{m_n}{e_n} = \frac{(K_p + K_I T) - K_p \beta}{1 - \beta} = G_c(\beta)$$

Closed-Loop TF:

$$\frac{C_n}{r_n} = \frac{G_c(\beta) G_p(\beta)}{1 + G_c(\beta) G_p(\beta)}$$

$$G_c(\beta) = \frac{(K_p + K_I T) - K_p \beta}{1 - \beta}$$

$$G_p(\beta) = \frac{K(1 - e^{-T/\tau})\beta}{1 - e^{-T/\tau}\beta}$$



Define: $K_p + K_E T \equiv C_1$
 $1 - e^{-T/\tau} \equiv C_2$
 $e^{-T/\tau} \equiv C_3$

$$\left. \begin{aligned} G_c(\beta) &= \frac{C_1 - K_p \beta}{1 - \beta} \\ G_p(\beta) &= \frac{K C_2 \beta}{1 - C_3 \beta} \end{aligned} \right\}$$

$$\frac{C_n}{r_n} = \frac{\left(\frac{C_1 - K_p \beta}{1 - \beta} \right) \left(\frac{K C_2 \beta}{1 - C_3 \beta} \right)}{1 + \left(\frac{C_1 - K_p \beta}{1 - \beta} \right) \left(\frac{K C_2 \beta}{1 - C_3 \beta} \right)}$$

$$\frac{C_n}{r_n} = \frac{(C_1 - K_p \beta)(K C_2 \beta)}{(1 - \beta)(1 - C_3 \beta) + (C_1 - K_p \beta)(K C_2 \beta)}$$

$$\frac{C_n}{r_n} = \frac{K C_2 (C_1 \beta - K_p \beta^2)}{1 + (-C_3 - 1 + K C_1 C_2) \beta + (C_3 - K_p K C_2) \beta^2}$$

Closed-Loop TF

Note: Input to plant is assumed constant between samples.

This Closed-Loop TF says:

$$C_n = (C_3 + 1 - K C_1 C_2) C_{n-1} + (-C_3 + K_p K C_2) C_{n-2} \\ + (K C_2 C_1) C_{n-1} + (-K C_2 K_p) C_{n-2}$$

$$\text{where } \begin{cases} C_1 = K_p + K_I T \\ C_2 = 1 - e^{-T/\tau} \\ C_3 = e^{-T/\tau} \end{cases}$$

Transform Methods

Laplace Transform

Laplace Variable $s \Leftrightarrow D = \frac{d}{dt}$ Differential Operator

Z Transform

Z Transform $z^{-1} \Leftrightarrow B$ Backshift Operator

Only when a zero-order hold is present at the input of a continuous system

PID Controller:

$$m_n - m_{n-1} = K_0 e_n + K_1 e_{n-1} + K_2 e_{n-2}$$

$$G_c(z) = \frac{K_0 + K_1 z^{-1} + K_2 z^{-2}}{1 - z^{-1}} = \frac{M(z)}{E(z)}$$

$$K_0 = K_p + K_I T + \frac{K_0}{T}$$

$$K_1 = -K_p - \frac{2K_0}{T}$$

$$K_2 = \frac{K_0}{T}$$

PI Controller:

$$\frac{m_n}{e_n} = \frac{(K_p + K_I T) - K_p \beta}{1 - \beta}$$

$$\frac{M(z)}{E(z)} = \frac{(K_p + K_I T) - K_p z^{-1}}{1 - z^{-1}}$$