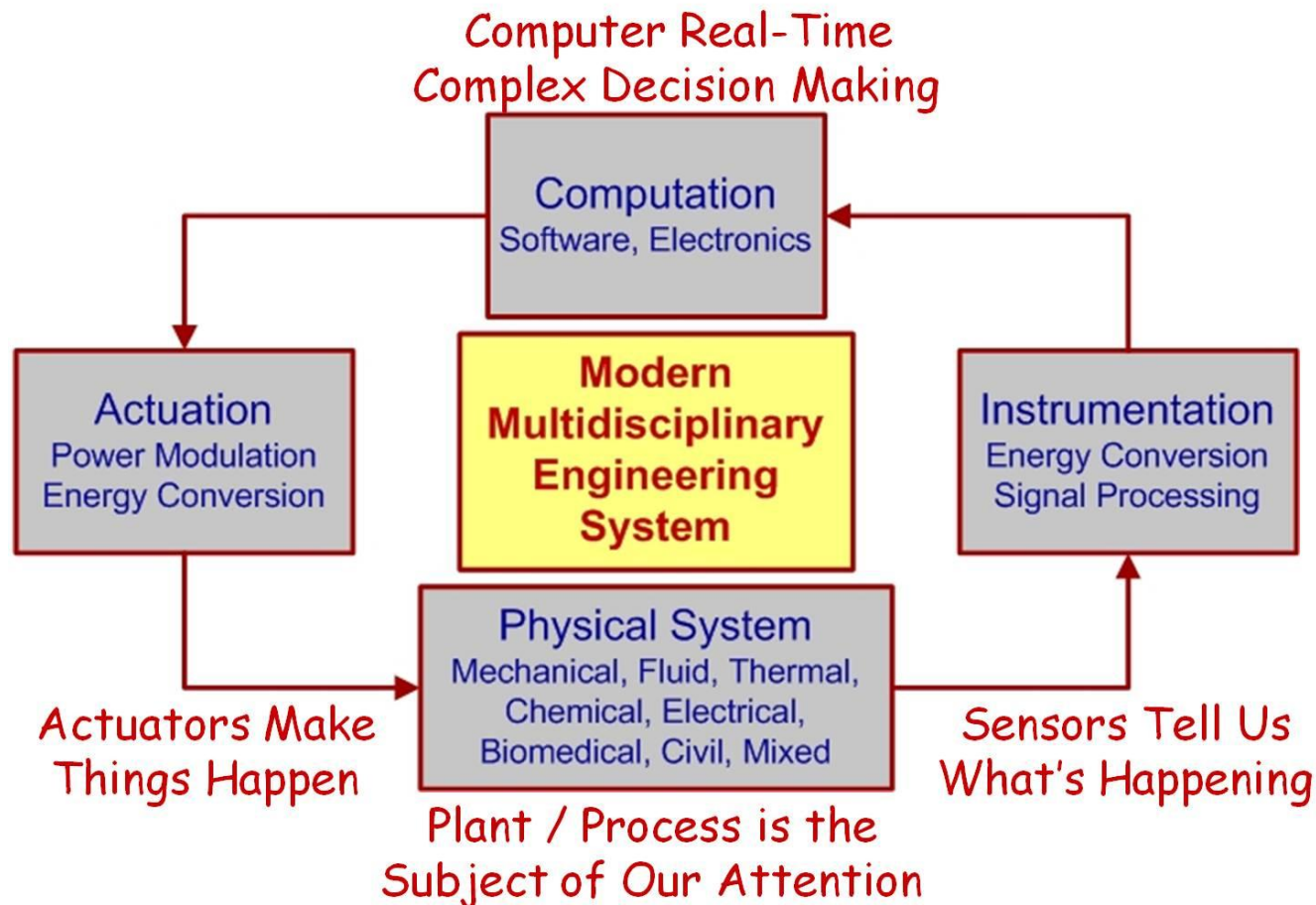


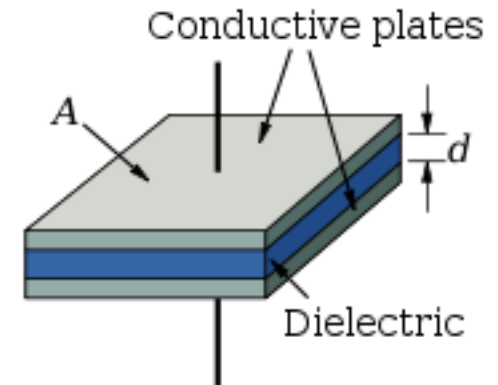
# Electrical Systems: Modeling, Analysis, Measurement, & Control

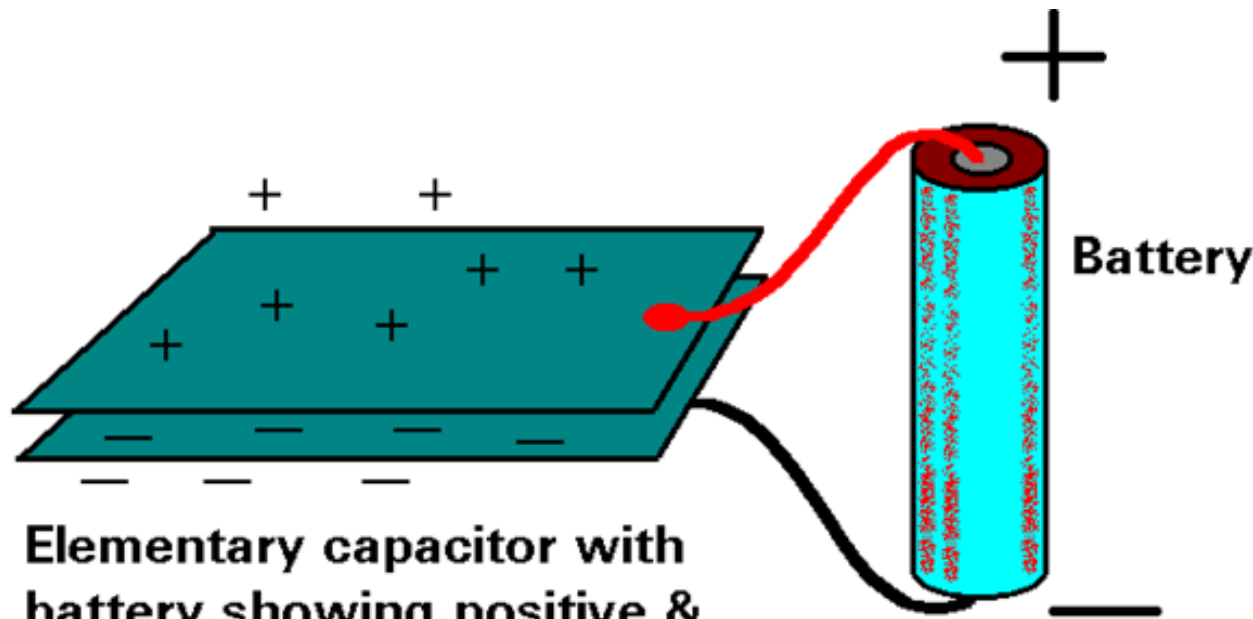


# Electrical System Topics

- Part 2
  - Capacitance
    - Physical Model
    - Mathematical Model
    - Impulse and Step Response
    - Frequency Response
    - Important Uses
  - RC Circuit (Low-Pass Filter – Anti-Aliasing Filter) System Investigation

## Capacitor

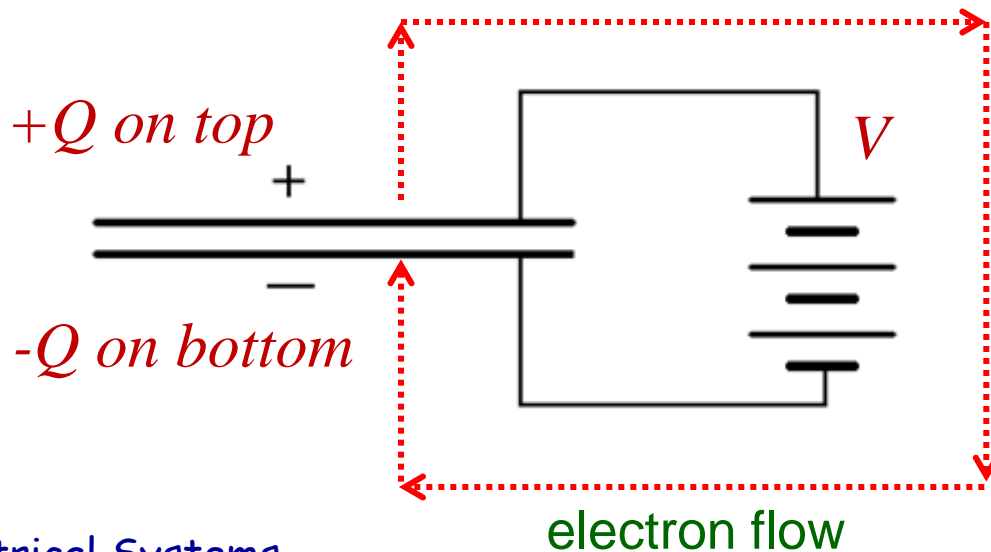




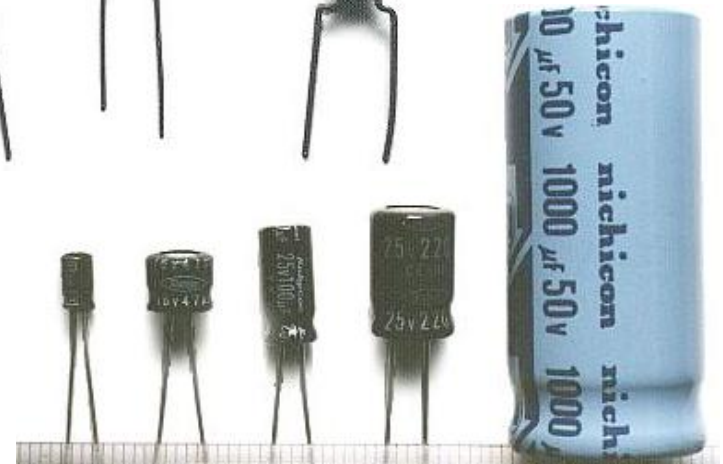
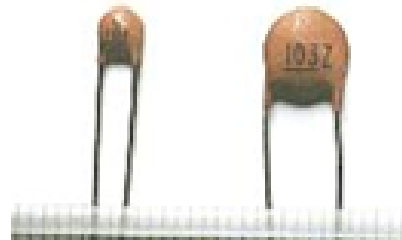
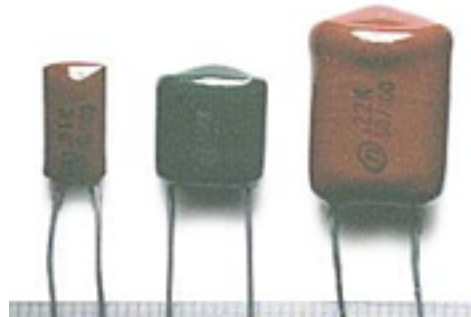
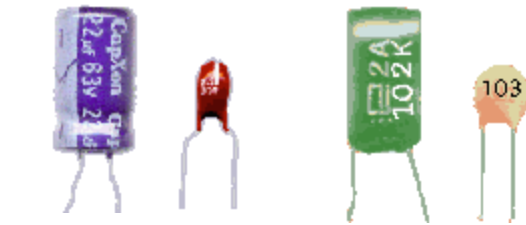
Elementary capacitor with battery showing positive & negative charges on its plates.

$$Q = CV$$

The battery causes electrons to be moved from the top plate of the capacitor to the bottom plate.



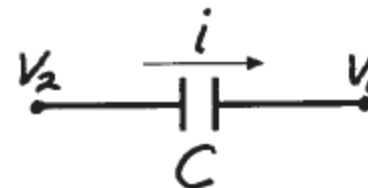
# Capacitors



# The Capacitance Element

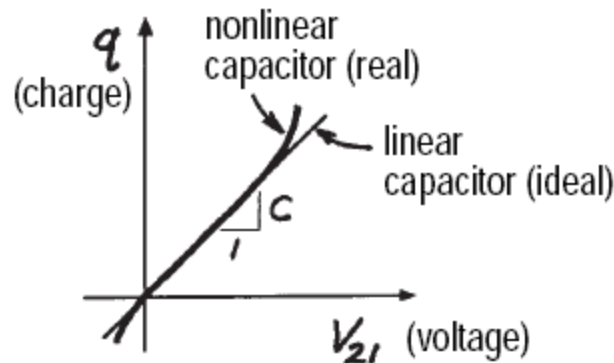
- This is another fundamental electrical element. Like a resistor, it is intentionally or unintentionally present in every real electrical system.
- Two conductors separated by a nonconducting medium (insulator or dielectric), that allows an electrostatic field to be established without allowing charge to flow between the two pieces of conducting material, form a capacitor.
- A capacitor stores electrical energy in its electrostatic field. In a pure and ideal capacitor, all of the energy stored in a capacitor can be retrieved and used.

$$C(\text{farads}) = \frac{q(\text{coulombs})}{e(\text{volts})}$$



## – Charging a Capacitor

- Process of removing charge from one conductor and placing an equal amount on the other.
  - The net charge of a capacitor is always zero and the “charge on a capacitor” refers to the magnitude of the charge on either conductor.
- In a pure and ideal capacitance element, the numerical value of  $C$  is absolutely constant for all values of  $q$  or  $e$ .
- Real capacitors exhibit some nonlinearity and are contaminated by the presence of resistance and/or inductance.



## Mathematical Model

$$e = \frac{1}{C}q \Rightarrow \frac{de}{dt} = \frac{1}{C} \frac{dq}{dt} = \frac{1}{C}i \Rightarrow i = C \frac{de}{dt}$$

$$de = \left( \frac{1}{C}i \right) dt \Rightarrow \int_{e_0}^e de = \frac{1}{C} \int_0^t (i) dt \Rightarrow e - e_0 = \frac{1}{C} \int_0^t (i) dt$$

$$\frac{i}{e}(D) = CD$$

$$\frac{e}{i}(D) = \frac{1}{CD}$$

Operational Transfer Functions

$$D = \frac{d}{dt}$$

$$De = \frac{de}{dt}$$

Differential  
Operator

- Energy Stored

- The pure and ideal capacitance stores in its electric field all the electrical energy supplied to it during the charging process and will give up all of this energy if completely discharged, say by connecting it to a resistor.
- The work done to transfer a charge  $dq$  through a potential difference  $e$  is  $(e)dq$ . The total energy stored by a charged capacitor is:

$$\int_0^q (e) dq = \int_0^q \left( \frac{q}{C} \right) dq = \frac{q^2}{2C} = \frac{Ce^2}{2}$$

- This is true irrespective of how the final voltage or charge was built up.
- There is no current “through” a capacitor; an equal amount of charge is taken from one plate and supplied to the other by way of the circuit external to the capacitor.



# Capacitance Element

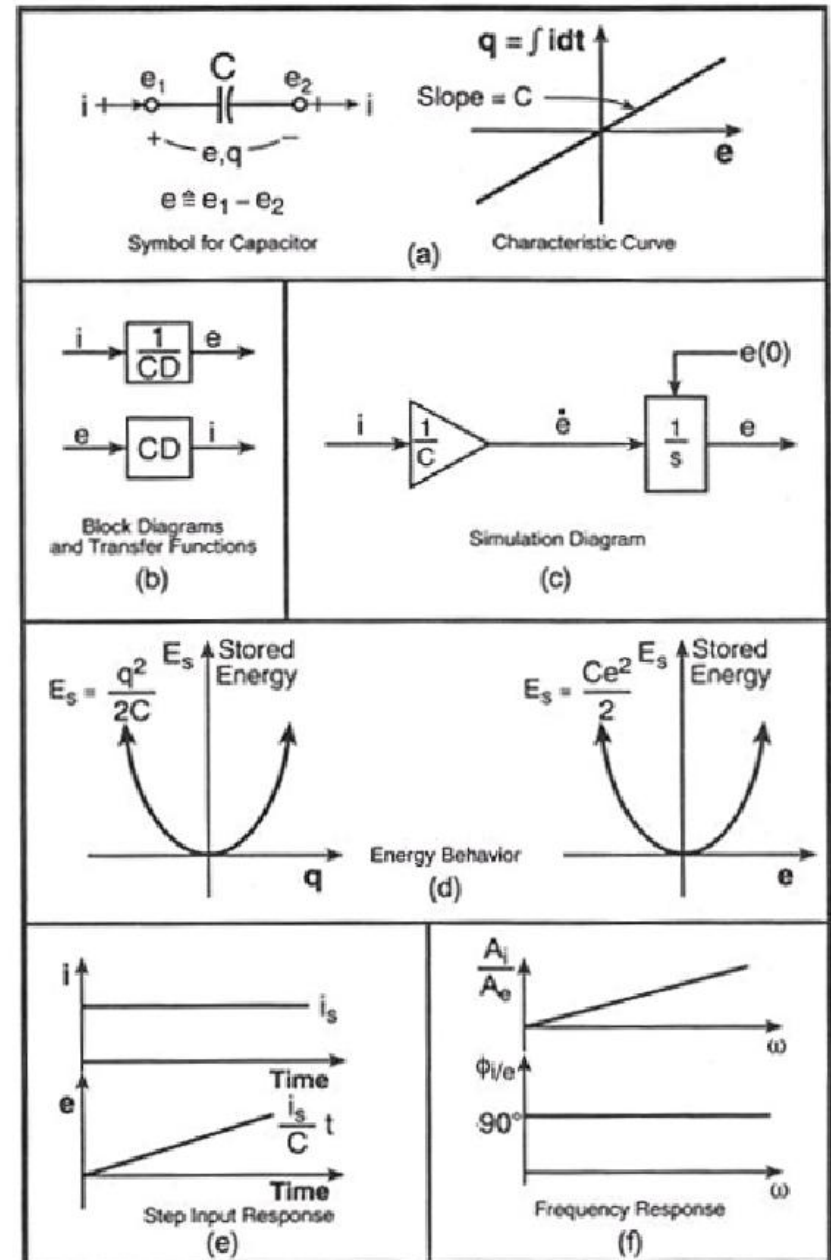
$$q = Ce$$

$$i = C \frac{de}{dt} = CDe$$

$$\frac{i}{e} (D) = CD$$

$$\frac{i}{e} (i\omega) = C(i\omega)$$

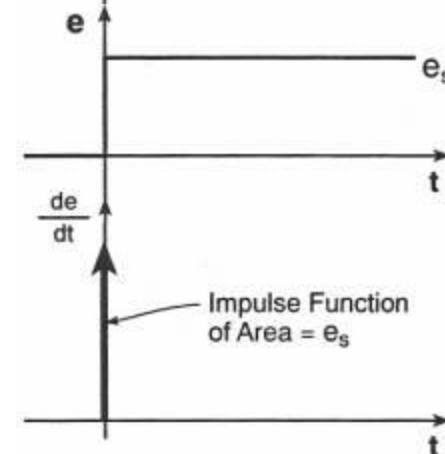
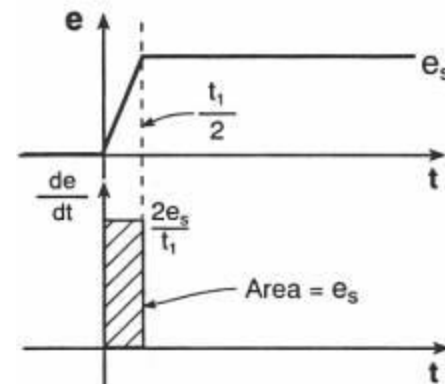
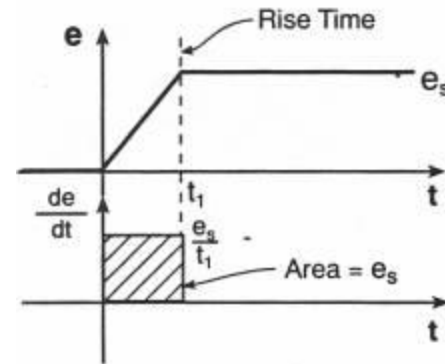
$$= C\omega \angle 90^\circ$$



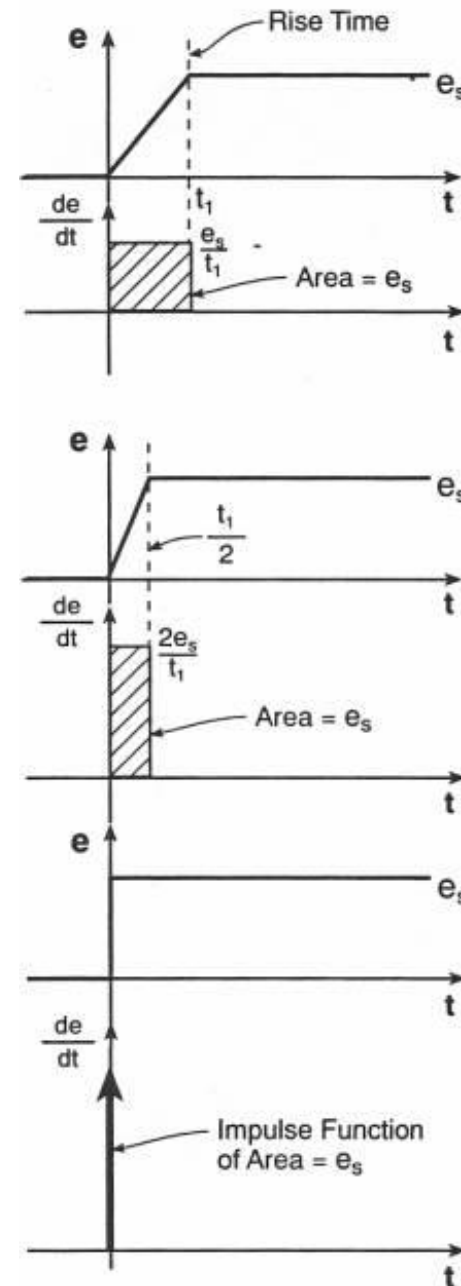
## Approximate and Exact Impulse Functions

If  $e_s = 1.0$  (unit step function),  
its derivative is the unit  
impulse function with a  
strength (or area) of one unit.

This “non-rigorous” approach  
does produce the correct result.



The impulse function is explained by the figure, where we approximate the step function by a terminated ramp and then let the rise time of the ramp approach zero. As we let the ramp get steeper and steeper, the magnitude of  $de/dt$  approaches infinity, and its duration approaches zero, but the area under it will always be  $e_s$ . If  $e_s = 1$  (a unit step function), its derivative is called the unit impulse function with an area or strength equal to one unit. The step function is the integral of the impulse function, or conversely, the impulse function is the derivative of the step function. When we multiply the impulse function by some number, we increase the “strength of the impulse”, but “strength” now means area, not height as it does for “ordinary” functions.



- A step input voltage produces a capacitor current of infinite magnitude and infinitesimal time duration. Real physical quantities are limited to finite values.
  - A true (instant rising) step voltage cannot be achieved.
  - A real capacitor has parasitic resistance and inductance which limit current and its rate of change.
  - Thus, a real capacitor will exhibit a short-lived (but not infinitesimal) and large (but not infinite) current spike.
- Impulse functions appear whenever we try to differentiate discontinuous functions.

- An impulse that has an infinite magnitude and zero duration is mathematical fiction and does not occur in physical systems. If, however, the magnitude of a pulse input to a system is very large and its duration is very short compared to the system's speed of response, then we can approximate the pulse input by an impulse function. The impulse input supplies energy to the system in an infinitesimal time.
- The step response of a component or system is the time response to a step input of some magnitude. The impulse response of a system is the derivative of the step response and is the time response to an impulse input of some strength.

## Step Response

## Capacitor

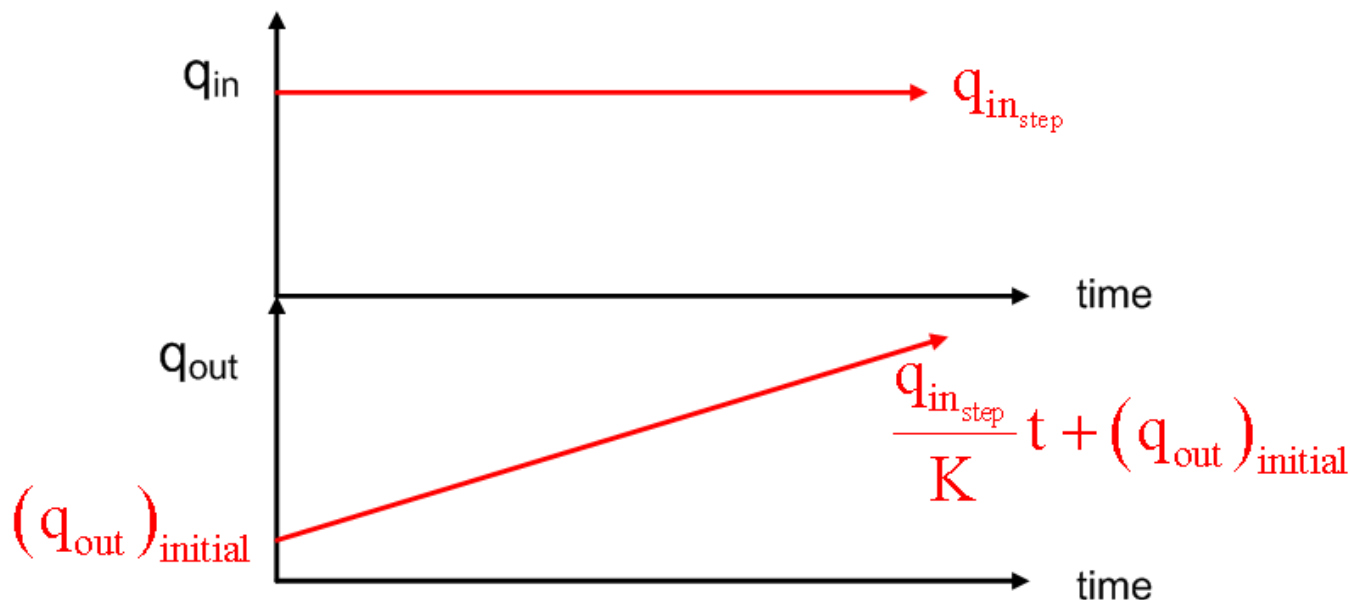
$$i = C \frac{de}{dt} = CDe$$

$$e = \frac{1}{CD} i$$

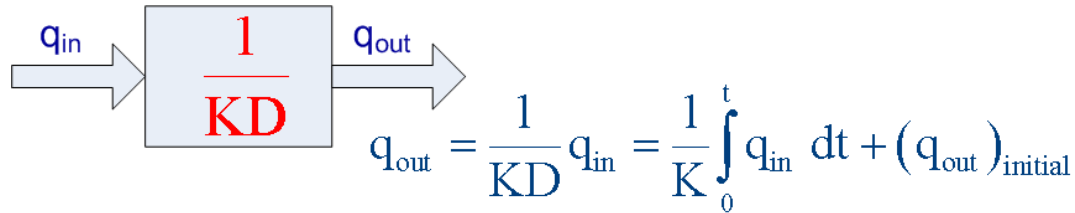
$$q_{in} = i$$
$$q_{out} = e$$



$$q_{out} = \frac{1}{KD} q_{in} = \frac{1}{K} \int_0^t q_{in} dt + (q_{out})_{initial}$$



# Frequency Response (Steady- State)



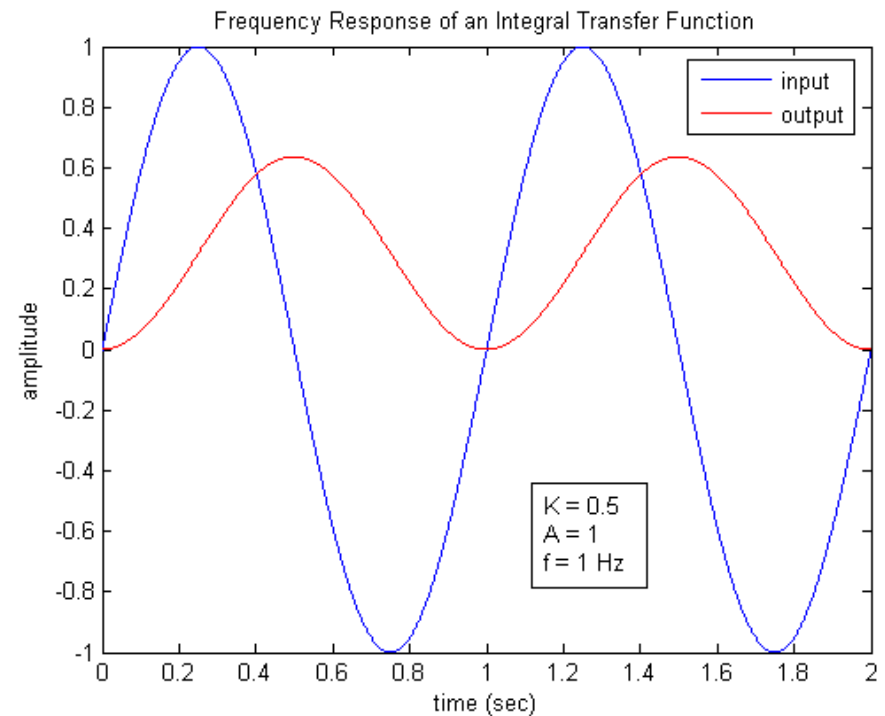
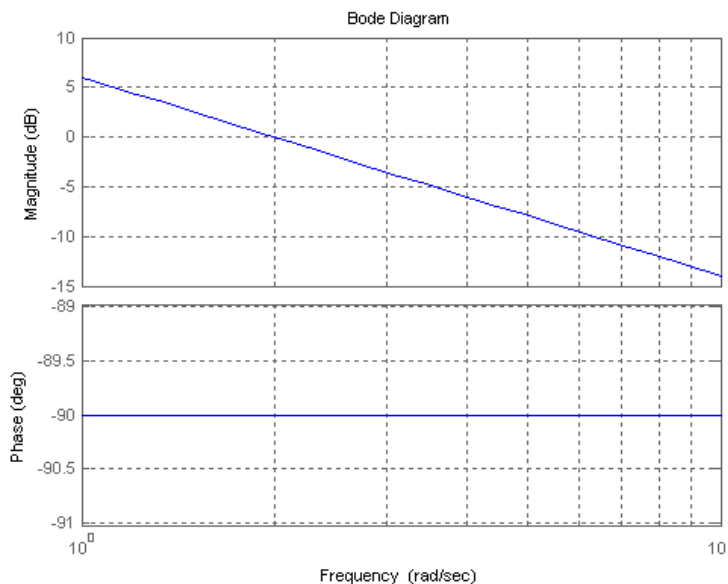
$$i = C \frac{de}{dt} = CDe$$

$$q_{in} = A \sin(\omega t)$$

$$q_{out} = \frac{-A}{K\omega} \cos(\omega t) + (q_{out})_{initial}$$

$$= \frac{A}{K\omega} \sin\left(\omega t - \frac{\pi}{2}\right) + (q_{out})_{initial}$$

$$e = \frac{1}{CD} i \quad \begin{matrix} q_{in} = i \\ q_{out} = e \end{matrix}$$



- Capacitors in Series and Parallel

- The total capacitance of capacitors connected in parallel is the sum of the individual capacitances.
- Capacitance is proportional to the area of the conducting plates. Placing capacitors in parallel is like increasing the area of the capacitor plates.

$$C_{\text{total}} = C_1 + C_2 + \cdots + C_n$$

- The total capacitance of capacitors connected in series is the reciprocal of the sum of the reciprocals of the individual capacitors.
- Capacitance is inversely proportional to the spacing between the conducting plates. Placing capacitors in series is like increasing the separation between the capacitor plates.

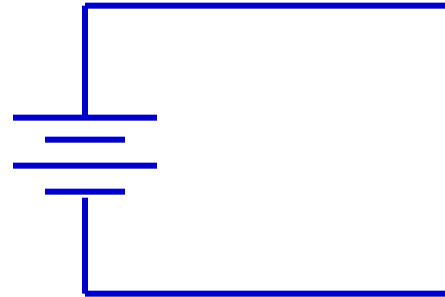
$$\frac{1}{C_{\text{total}}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}$$



Resistors behave the same at all frequencies.  
Capacitors do not.

Will carry any current the source can produce until the wire burns up.

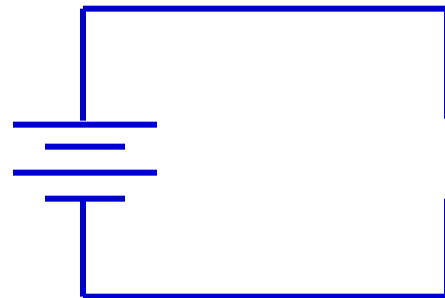
Short Circuit



Capacitor  
at high frequency

Will carry no current no matter how large the voltage is, unless arcing occurs.

Open Circuit



Capacitor  
at low frequency

- Capacitor Usage

- Large capacitors are used in power supply circuits to store charge, or energy, for delivery at a later time.
- They are used to filter out a 60 Hz ripple.
- The function of a bypass capacitor is to ensure that the *dc* component of a signal appears on some circuit element but that the *ac* component is shorted out or bypassed around the element. They are used to prevent high-frequency noise on the dc power line in modern digital electronics from entering into the logic via the power leads.
- A blocking or coupling capacitor blocks the dc component of a signal from propagating to another section of a circuit while allowing ac signals to get through.

- Capacitors are used in frequency discrimination, or timing, circuits. RC and LC networks are used to create shaped frequency responses.
- Capacitors are used in integrating circuits for measuring charge or in analog-to-digital converters.
- Capacitor Properties
  - In a capacitor consisting of two flat conductors separated by an insulator, capacitance is proportional to the area of the conductors and the dielectric constant, and inversely proportional to the spacing between the two conductors. Therefore to get large capacitance one needs large area, large dielectric constant, or small gap spacing. All three are used in real capacitors.

- Dielectric strength is an important parameter of the dielectric in a capacitor, i.e., how much voltage can be applied across it before it breaks down in some way and starts to conduct. Some dielectrics are healing, i.e., after they break down and the discharge is terminated, the dielectric reforms and is essentially unchanged.
- One of the two conductors in any rolled or multilayer capacitor is the outer conductor while the other conductor is shielded by the outer one. The outer conductor can pick up or transmit signals as an antenna and should be connected to ground.

- Some leakage current will flow through the dielectric in any capacitor and this is expressed in terms of an equivalent resistance ( $10^6$  to  $10^{12}$  ohms).
- Some of the energy put into charging a capacitor is lost; it appears as heat in the dielectric. There is dielectric loss of the capacitor and the amount increases with frequency and depends on the type of insulator.
- There are two time constants in a capacitive circuit.
  - The energy stored in the electric field between the two plates and most of the energy stored in the polarization of the dielectric can be removed as fast as the RC time constant of the circuit will allow the capacitor to discharge.

- Some of the polarization energy is released on a time scale determined by atomic processes in the material; this can have time constants measured in milliseconds to days. Thus a large capacitor that has a highly polarized dielectric can be discharged and then left with its leads open; some time later, it is found that there is some voltage between the two terminals, as some of the energy stored in the polarization has been slowly returned to the electric field.
- Any real capacitor will have some parasitic inductance. Any capacitor will look inductive at a high enough frequency. To have a capacitor that will operate from low to extremely high frequencies, use several capacitors in parallel.

- Capacitor Specification

- Value
- Tolerance which gives the possible error in the nominal value of the capacitor. In general, the tolerances on capacitors are large and frequently asymmetrical.
- Voltage Rating (short-term and working) is the maximum voltage that can be applied to a capacitor without breakdown. It depends on the dielectric thickness and material.
- Insulation resistance is a measure of the ohmic resistance of the dielectric layer of the capacitor.
- Dissipation or power factor is a measure of the energy loss in the capacitor due to resistive leakage and dielectric loss.

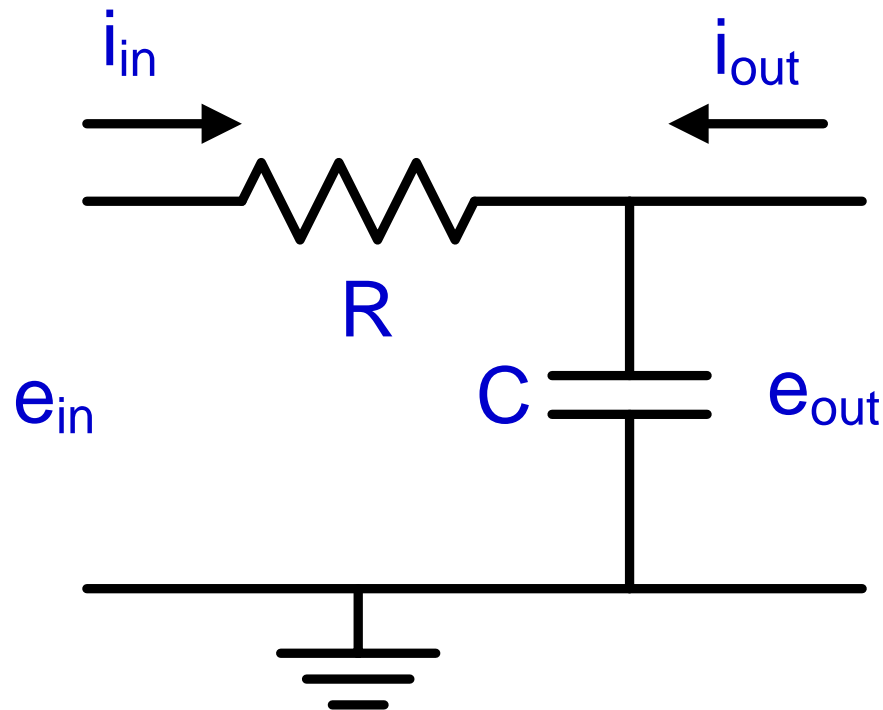
- The **quality factor**, dependent on the measurement frequency, indicates the energy loss in a capacitor. A near-perfect capacitor, with no losses, has a quality factor approaching infinity.
- The properties of the dielectric, and hence the capacitance of a capacitor, will change as a function of temperature. The **temperature coefficient** will indicate this dependence.
- The **voltage coefficient** indicates how the capacitance of a capacitor decreases as the voltage increases.
- **Aging specification** indicates the changes in the capacitor as a function of time.
- Capacitor Markings
  - Measure It!



- Fixed Capacitor Types and Properties
  - Two main classes of fixed capacitors: electrolytic and nonelectrolytic.
  - Most electrolytic capacitors are polarized and will have some markings to indicate which lead must be the positive one.
  - The two main types of electrolytic capacitors are:  
Aluminum and Tantalum
    - Aluminum electrolytic capacitors are the most common electrolytic capacitors in use. They have capacitances ranging from 1 to  $10^6$   $\mu\text{F}$  and in voltage ranges from 100 to 700 V.

- Tantalum electrolytic capacitors are smaller than aluminum for equivalent ratings. They have better characteristics in all respects, cost more, and have longer life expectancy. They have a capacitance range of 0.1 to 1000  $\mu\text{F}$  and a voltage range of 3 to 150 V.
- Nonelectrolytic capacitors are made of a variety of new materials, e.g., polypropylene, polyimide, polystyrene, polycarbonate, polyester, paper, mica, glass, and ceramic.

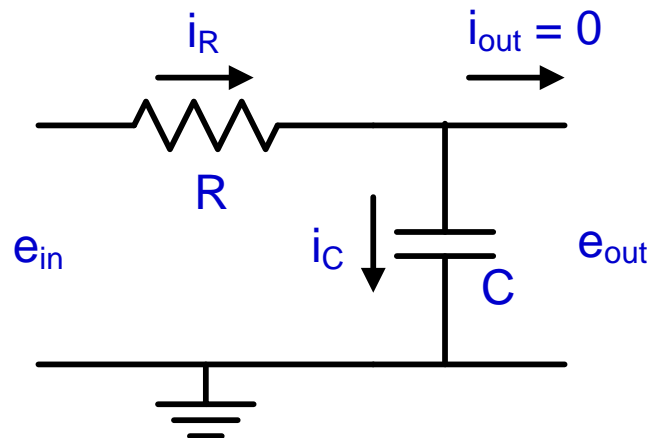
# Physical System for Investigation



## RC Circuit Electrical System

# Physical Modeling

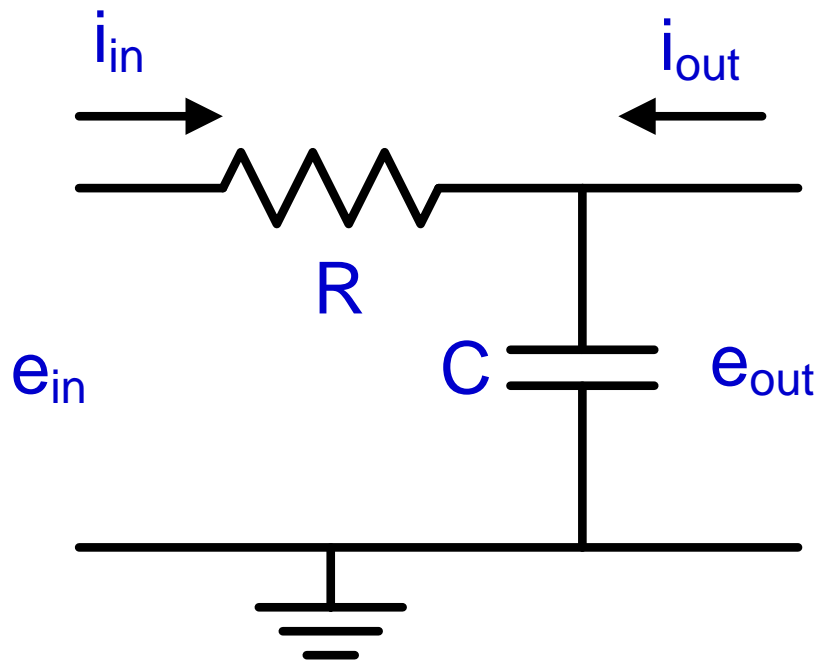
- Simplifying Assumptions
  - Resistors and Capacitors are pure and ideal.
  - Voltage sources are ideal and supply the intended voltage to the circuit no matter how much current (and thus power) this might require.
  - Measuring devices are ideal and do not load the circuits by drawing any current.



# Model Parameter Identification

- Measure resistor and capacitor component values using the DMM. Note tolerances.
- RC Circuit
  - 15 k $\Omega$  nominal (5% tolerance), 14.986 k $\Omega$  measured
  - 0.01  $\mu$ F nominal, 9.85 nF = 0.00985  $\mu$ F measured

# Mathematical Modeling



## Apply to Physical Model

- KVL and KCL
- Constitutive Equations
  - Capacitor
  - Resistor

## Complete Mathematical Model

$$\begin{bmatrix} e_{in} \\ i_{in} \end{bmatrix} = \begin{bmatrix} RCD + 1 & -R \\ CD & -1 \end{bmatrix} \begin{bmatrix} e_{out} \\ i_{out} \end{bmatrix}$$

$$\begin{bmatrix} e_{out} \\ i_{out} \end{bmatrix} = \begin{bmatrix} 1 & -R \\ CD & -(RCD + 1) \end{bmatrix} \begin{bmatrix} e_{in} \\ i_{in} \end{bmatrix}$$

## Basic Component (R and C) Equations

(Constitutive Equations)

$$e_{\text{in}} - e_{\text{out}} = iR$$

$$i = C \frac{de_{\text{out}}}{dt}$$

1<sup>st</sup>-Order Linear

Constant-Coefficient ODE

KCL  $i_R = i_C + i_{\text{out}}$

$$i_R = i_C + 0$$

$$i_R = i_C$$

$$\frac{e_{\text{in}} - e_{\text{out}}}{R} = C \frac{de_{\text{out}}}{dt}$$

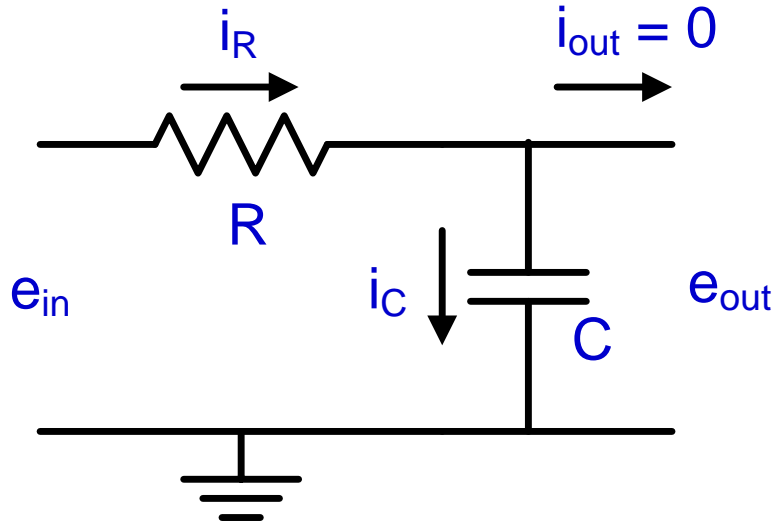
$$RC \frac{de_{\text{out}}}{dt} + e_{\text{out}} = e_{\text{in}}$$

$$K = 1$$

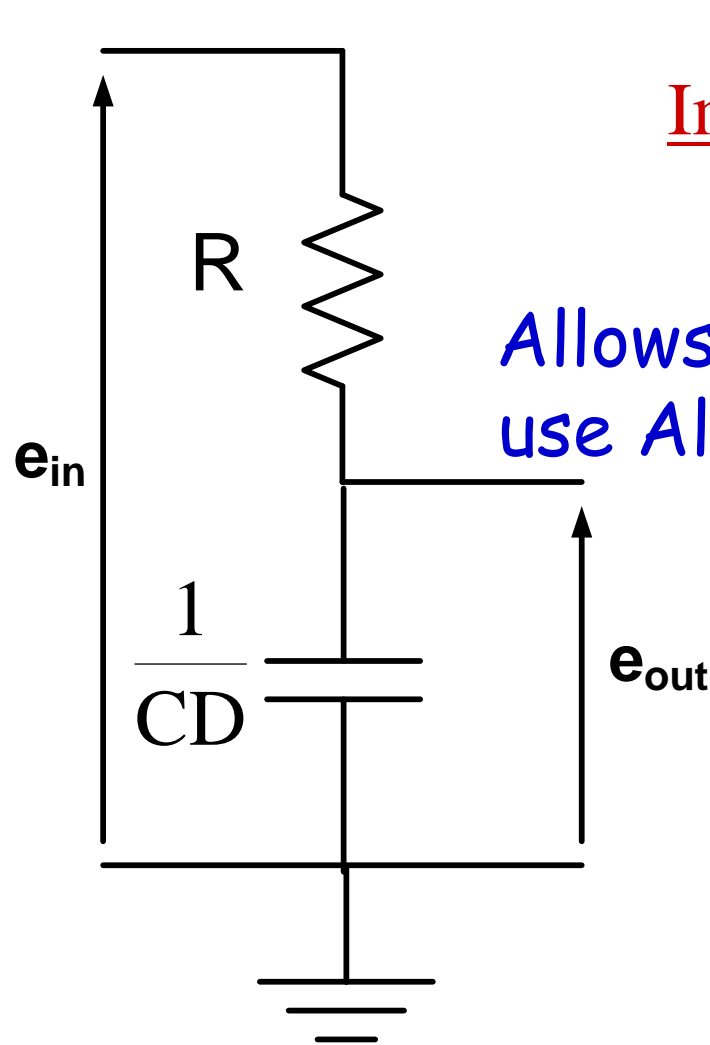
$$\tau \frac{de_{\text{out}}}{dt} + e_{\text{out}} = Ke_{\text{in}}$$

$$\tau = RC$$

General Form



# Preferred Approach: Impedance + Voltage Divider



Impedance

$$e / i$$

Allows us to  
use Algebra

$$e = iR$$

$$\frac{e}{i} = R$$

$$i = C \frac{de}{dt} = CD e$$

$$\frac{e}{i} = \frac{1}{CD}$$

$$D = \frac{d}{dt} \text{ Differential Operator}$$

$$\frac{e_{out}}{e_{in}} = \frac{\frac{1}{CD}}{R + \frac{1}{CD}} = \frac{1}{RCD + 1}$$

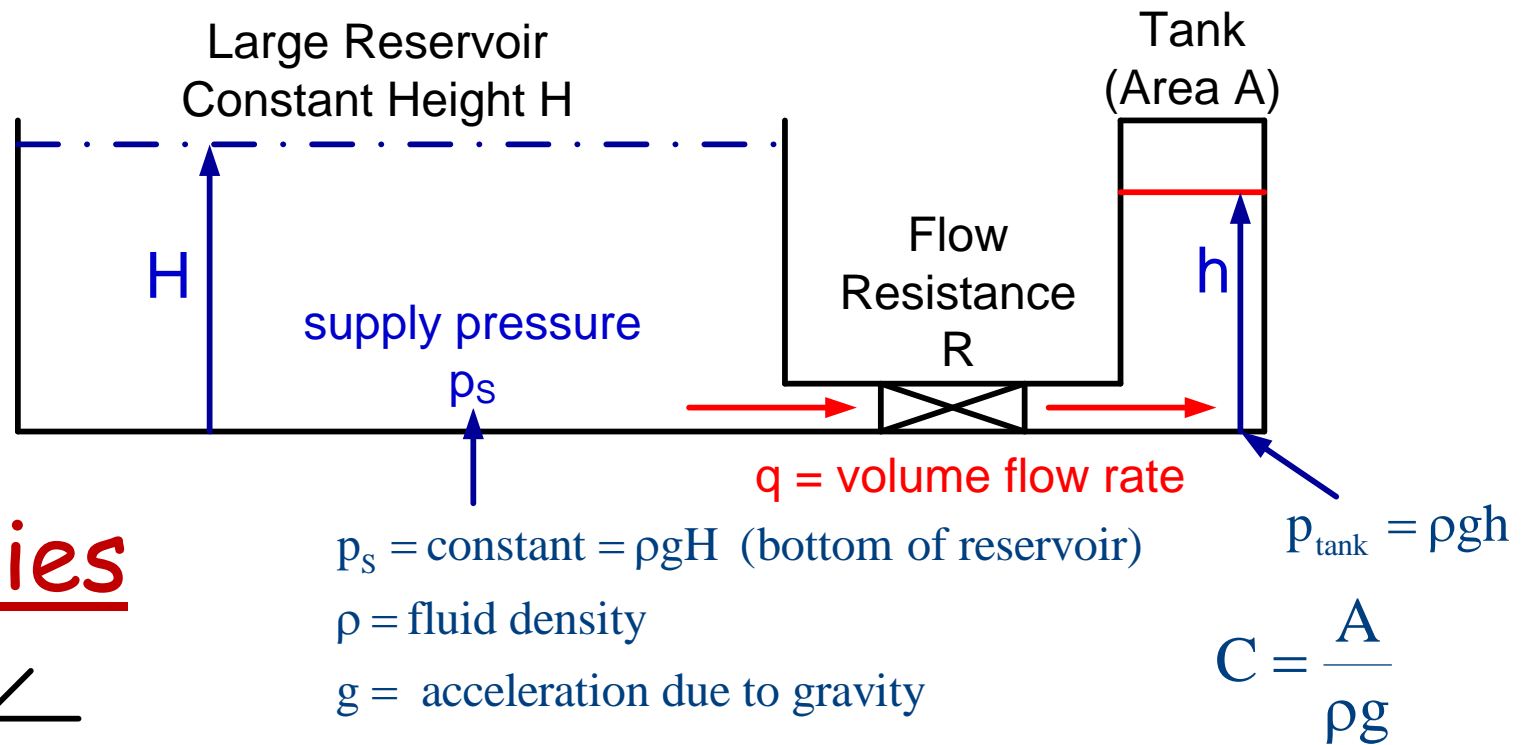
Transfer  
Function

$$(RCD + 1)e_{out} = (1)e_{in}$$

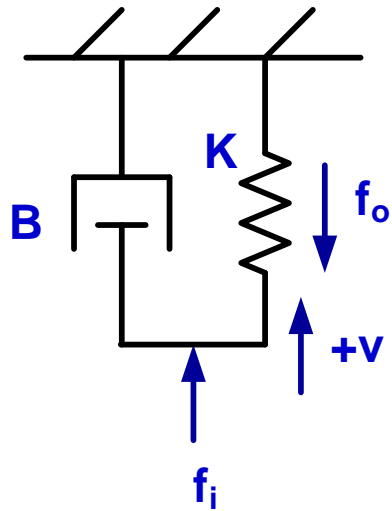
$$(RCD)e_{out} + e_{out} = e_{in}$$

$$RC \frac{de_{out}}{dt} + e_{out} = e_{in}$$





## Analogies



$$\frac{B}{K} \frac{df_o}{dt} + f_o = f_i$$

$$RC \frac{dh}{dt} + h = H$$

$$\tau \frac{de_{\text{out}}}{dt} + e_{\text{out}} = K e_{\text{in}}$$

# Analytical Solution

## 1<sup>st</sup>-Order, Linear, Constant-Coefficient ODE

$$\tau \frac{de_{\text{out}}}{dt} + e_{\text{out}} = Ke_{\text{in}}$$

- The classical operator method of solution is a three-step procedure:
  - Find the complimentary (homogeneous) solution for the equation with the input equal to zero.
  - Find the particular solution with the input present.
  - Get the complete solution, i.e., the sum of the complimentary and particular solutions, and evaluate the constants of integration in the complimentary solution by applying known initial conditions.

## Step 1

Same regardless of input, as  $e_{in} = 0$ .

$$\tau \frac{de_{out}}{dt} + e_{out} = Ke_{in} \Rightarrow \tau D e_{out} + e_{out} = Ke_{in}$$

$$(\tau D + 1)e_{out} = Ke_{in} \Rightarrow \tau D + 1 = 0$$

  
characteristic equation

$$D = \frac{-1}{\tau} \Rightarrow (e_{out})_c = C_1 e^{Dt} = C_1 e^{\frac{-t}{\tau}}$$

$$(e_{out})_c = C_1 e^{\frac{-t}{\tau}}$$

Constant  $C_1$  is determined from the initial condition once the complete solution is formed.

## Step 2

$$\tau \frac{de_{\text{out}}}{dt} + e_{\text{out}} = Ke_{\text{in}}$$

Step Input  
 $e_{\text{in}} = \text{constant}$

$$(e_{\text{out}})_p = C_2 \Rightarrow \tau \frac{d(e_{\text{out}})_p}{dt} + (e_{\text{out}})_p = Ke_{\text{in}}$$

$$C_2 = Ke_{\text{in}} \Rightarrow (e_{\text{out}})_p = Ke_{\text{in}}$$

$$(e_{\text{out}})_p = Ke_{\text{in}}$$

## Step 3

$$e_{\text{out}} = (e_{\text{out}})_c + (e_{\text{out}})_p = C_1 e^{\frac{-t}{\tau}} + Ke_{\text{in}}$$

$$\text{I.C. } t = 0, e_{\text{out}} = 0 \Rightarrow C_1 = -Ke_{\text{in}}$$

## Complete Solution

$$e_{\text{out}} = Ke_{\text{in}} \left( 1 - e^{\frac{-t}{\tau}} \right)$$

$$\text{At } t = \tau, e_{\text{out}} = Ke_{\text{in}} (1 - e^{-1}) = (0.632) Ke_{\text{in}}$$

# Sine Input

## Step 2

$$\tau \frac{de_{\text{out}}}{dt} + e_{\text{out}} = Ke_{\text{in}} \Rightarrow e_{\text{in}} = A \sin(\omega t)$$

$$(e_{\text{out}})_p = C_2 A \sin(\omega t + \phi) \Rightarrow \tau \frac{d(e_{\text{out}})_p}{dt} + (e_{\text{out}})_p = Ke_{\text{in}}$$

$$\tau C_2 A \omega \cos(\omega t + \phi) + C_2 A \sin(\omega t + \phi) = KA \sin(\omega t)$$

$$\tau C_2 A \omega [\cos(\omega t) \cos(\phi) - \sin(\omega t) \sin(\phi)] +$$

$$C_2 A [\sin(\omega t) \cos(\phi) + \cos(\omega t) \sin(\phi)] = KA \sin(\omega t)$$

$$[\tau C_2 A \omega \cos(\phi) + C_2 A \sin(\phi)] \cos(\omega t) +$$

$$[-\tau C_2 A \omega \sin(\phi) + C_2 A \cos(\phi)] \sin(\omega t) = KA \sin(\omega t)$$

$$\tau C_2 A \omega \cos(\phi) + C_2 A \sin(\phi) = 0$$

$$-\tau C_2 A \omega \sin(\phi) + C_2 A \cos(\phi) = KA$$

$$\begin{bmatrix} \tau C_2 \omega & C_2 \\ C_2 & -\tau C_2 \omega \end{bmatrix} \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix} = \begin{bmatrix} 0 \\ K \end{bmatrix}$$

$$\cos(\phi) = \frac{\begin{vmatrix} 0 & C_2 \\ K & -\tau C_2 \omega \end{vmatrix}}{\begin{vmatrix} \tau C_2 \omega & C_2 \\ C_2 & -\tau C_2 \omega \end{vmatrix}} = \frac{-C_2 K}{-C_2^2 (1 + \tau^2 \omega^2)} = \frac{K}{C_2 (1 + \tau^2 \omega^2)}$$

$$\sin(\phi) = \frac{\begin{vmatrix} \tau C_2 \omega & 0 \\ C_2 & K \end{vmatrix}}{\begin{vmatrix} \tau C_2 \omega & C_2 \\ C_2 & -\tau C_2 \omega \end{vmatrix}} = \frac{K \tau C_2 \omega}{-C_2^2 (1 + \tau^2 \omega^2)} = \frac{-K \tau \omega}{C_2 (1 + \tau^2 \omega^2)}$$

$$\cos(\phi) = \frac{K}{C_2(1 + \tau^2\omega^2)}$$

$$\sin(\phi) = \frac{-K\tau\omega}{C_2(1 + \tau^2\omega^2)}$$

$$\frac{\sin(\phi)}{\cos(\phi)} = \tan(\phi) = -\tau\omega$$

$$\sin^2(\phi) + \cos^2(\phi) = 1 = \frac{(-K\tau\omega)^2}{C_2^2(1 + \tau^2\omega^2)^2} + \frac{K^2}{C_2^2(1 + \tau^2\omega^2)^2}$$

$$C_2^2(1 + \tau^2\omega^2)^2 = K^2(1 + \tau^2\omega^2)$$

$$C_2 = \frac{K}{\sqrt{1 + \tau^2\omega^2}}$$

$$e_{\text{in}} = A \sin(\omega t)$$

$$(e_{\text{out}})_p = C_2 A \sin(\omega t + \phi)$$

# Frequency Response

Steady-State Response of the System to an Input Sine Wave at a Particular Frequency

$$\tau \frac{de_{\text{out}}}{dt} + e_{\text{out}} = Ke_{\text{in}}$$
$$(e_{\text{out}})_{\text{ss}} = C_2 A \sin(\omega t + \phi) \left\{ \begin{array}{l} e_{\text{in}} = A \sin(\omega t) \\ \tan(\phi) = -\tau\omega \\ C_2 = \frac{K}{\sqrt{1 + \tau^2 \omega^2}} \end{array} \right.$$

The complimentary solution will decay to zero with time. The particular solution is the steady-state solution. The input is a sine wave and the steady-state output is a sine wave with the same frequency, but with a frequency-dependent amplitude and a frequency-dependent phase angle.

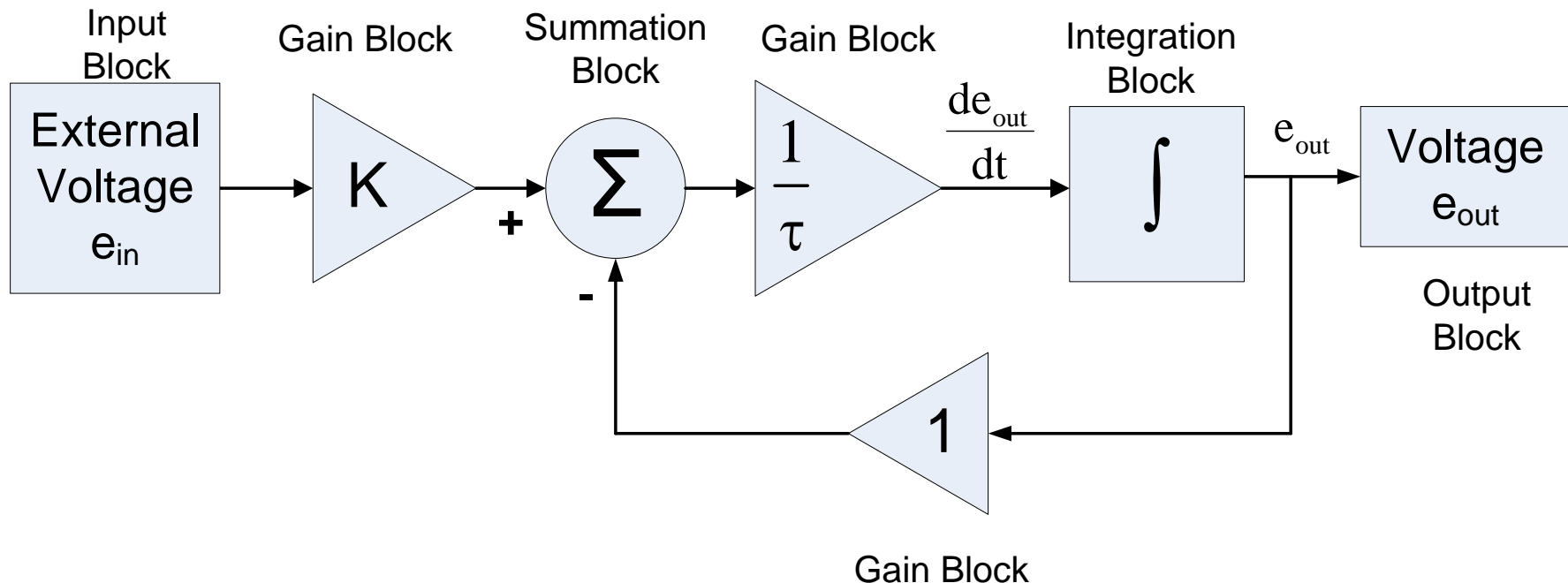


# 1<sup>st</sup> – Order System Block Diagram RC Electrical System

$$\tau = RC \quad K = 1$$

$$\tau \frac{de_{out}}{dt} + e_{out} = Ke_{in}$$

$$\frac{de_{out}}{dt} = \frac{1}{\tau} [Ke_{in} - e_{out}]$$



## MatLab / Simulink Numerical Solution

- Plotting Numerical Data

- Engineers are well known for their ability to plot many curves of experimental data and to extract all sorts of significant facts from these curves.
- The better one understands the physical phenomena involved in a certain experiment, the better is one able to extract a wide variety of information from graphical displays of experimental data.
- **Understand The Physical Processes Behind The Data!**
- When data may be approximated by a straight line, the analytical relation is easy to obtain; but when almost any other functional variation (e.g., exponential, polynomial, complex logarithmic) is present, difficulties are usually encountered.
- It is convenient to try to plot data in such a form that a straight line will be obtained for certain types of functional relationships.

# Analytical Solution to a Step Input

$$e_{\text{out}} = Ke_{\text{in}} \left( 1 - e^{-\frac{t}{\tau}} \right)$$

$$\frac{e_{\text{out}}}{Ke_{\text{in}}} = 1 - e^{-\frac{t}{\tau}}$$

$$1 - \frac{e_{\text{out}}}{Ke_{\text{in}}} = e^{-\frac{t}{\tau}}$$

$$\frac{Ke_{\text{in}} - e_{\text{out}}}{Ke_{\text{in}}} = e^{-\frac{t}{\tau}}$$

$$\frac{Ke_{\text{in}}}{Ke_{\text{in}} - e_{\text{out}}} = e^{\frac{t}{\tau}}$$

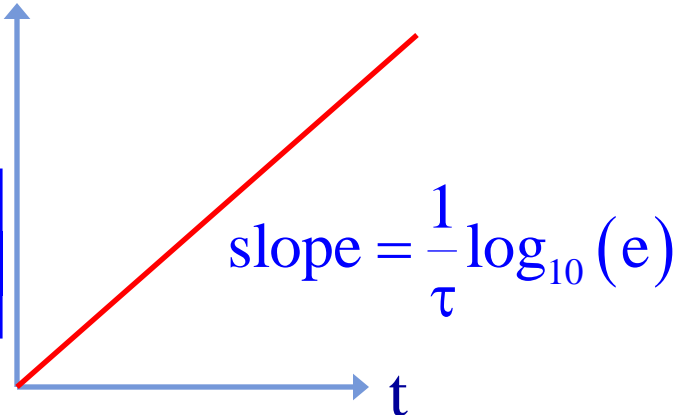
$$\frac{Ke_{\text{in}}}{Ke_{\text{in}} - e_{\text{out}}} = e^{\frac{t}{\tau}}$$

$$\log_{10} \left[ \frac{Ke_{\text{in}}}{Ke_{\text{in}} - e_{\text{out}}} \right] = \frac{t}{\tau} \log_{10} (e)$$

$$\log_{10} \left[ \frac{Ke_{\text{in}}}{Ke_{\text{in}} - e_{\text{out}}} \right] = \left[ \frac{1}{\tau} \log_{10} (e) \right] t$$

$$\log_{10} \left[ \frac{Ke_{\text{in}}}{Ke_{\text{in}} - e_{\text{out}}} \right]$$

$$\begin{cases} \tau \frac{de_{\text{out}}}{dt} + e_{\text{out}} = Ke_{\text{in}} \\ e_{\text{out}} = Ke_{\text{in}} \left( 1 - e^{-\frac{t}{\tau}} \right) \end{cases}$$



- This approach gives a more accurate value of  $\tau$  since the best line through all the data points is used rather than just two points, as in the 63.2% method. Furthermore, if the data points fall nearly on a straight line, we are assured that the instrument is behaving as a first-order type. If the data deviate considerably from a straight line, we know the system is not truly first order and a  $\tau$  value obtained by the 63.2% method would be quite misleading.
- An even stronger verification (or refutation) of first-order dynamic characteristics is available from frequency-response testing. If the system is truly first-order, the amplitude ratio follows the typical low- and high-frequency asymptotes (slope 0 and  $-20$  dB/decade) and the phase angle approaches  $-90^\circ$  asymptotically.

- If these characteristics are present, the numerical value of  $\tau$  is found by determining  $\omega$  (rad/sec) at the breakpoint and using  $\tau = 1/\omega_{\text{break}}$ . Deviations from the above amplitude and/or phase characteristics indicate non-first-order behavior.

# Mathematical Analysis and Prediction

$$RC \frac{de_{out}}{dt} + e_{out} = e_{in}$$

$$RC(D e_{out}) + e_{out} = e_{in}$$

$$(RCD + 1)e_{out} = e_{in}$$

$$\frac{e_{out}}{e_{in}} = \frac{1}{RCD + 1} = \frac{K}{\tau D + 1}$$

$K = 1 = \text{Steady-State Gain}$

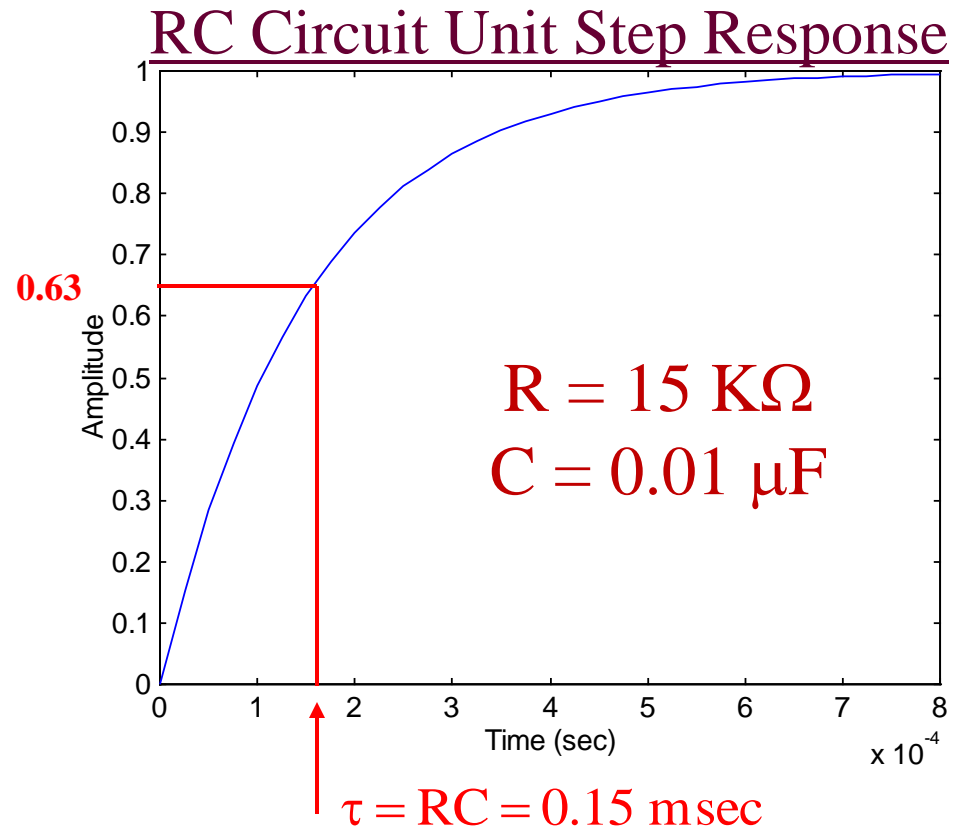
$\tau = RC = \text{Time Constant}$

$K = 1; \tau = 0.15 \text{E} - 3;$

MatLab Commands

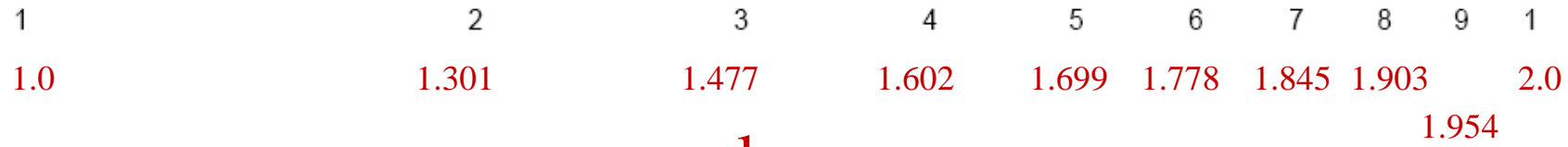
$\text{RC\_System} = \text{tf}(K, [\tau \ 1])$

$\text{step}(\text{RC\_System})$



- **Time Constant  $\tau$** 
  - Time it takes the step response to reach 63% of the steady-state value,  $Ke_{in}$ .
- **Rise Time  $T_r = 2.2 \tau$** 
  - Time it takes the step response to go from 10% to 90% of the steady-state value,  $Ke_{in}$ .
- **Delay Time  $T_d = 0.69 \tau$** 
  - Time it takes the step response to reach 50% of the steady-state value,  $Ke_{in}$ .
- **Steady-State Value**
  - The steady-state value of the response is  $Ke_{in}$  and at  $4\tau$  seconds (4 time constants), the response has reached 98% of the steady-state value; for all practical purposes, this is steady state.

1.0      1.1      1.2      1.3      1.4      1.5      1.6      1.7      1.8      1.9      2.0



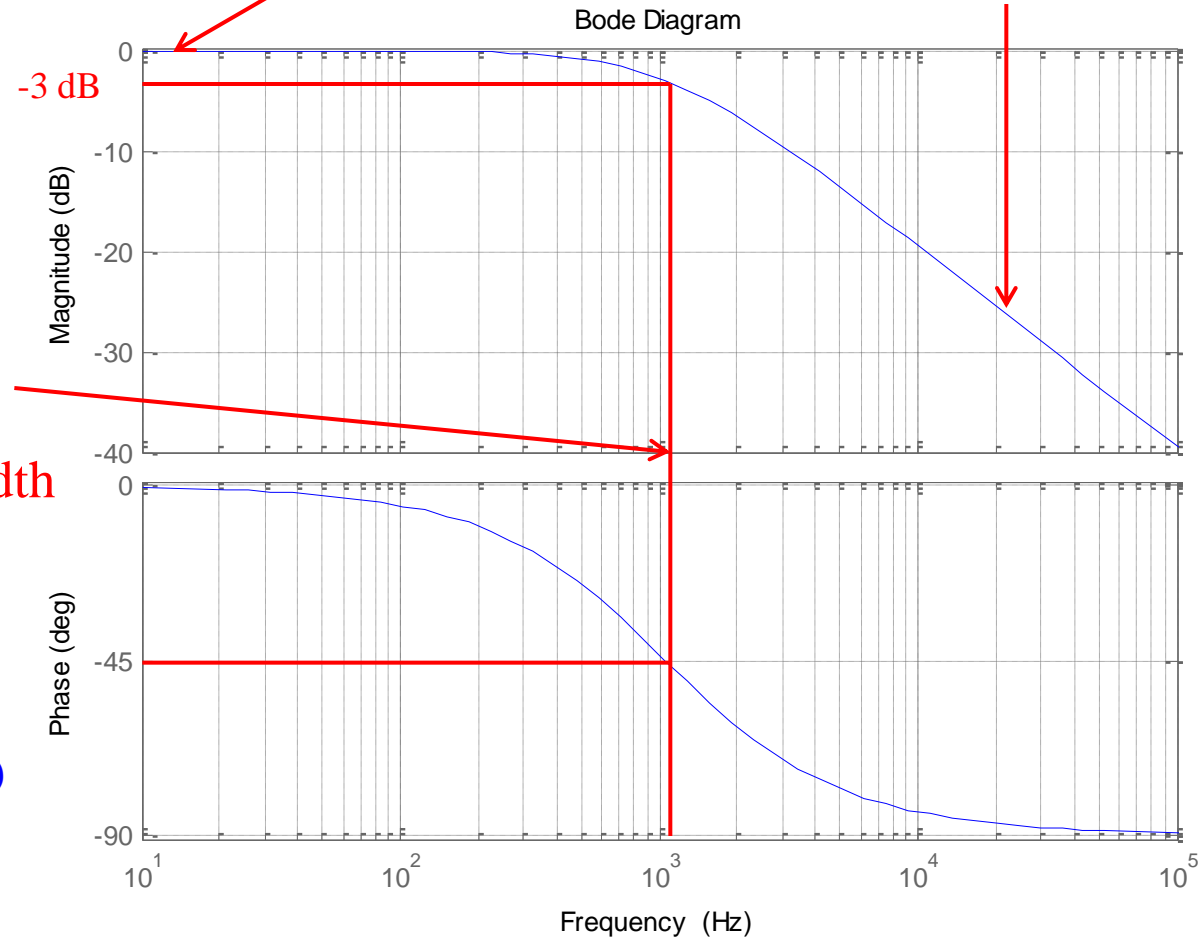


# RC Circuit Frequency Response

DC Value = K

$$(x) \text{ dB} = 20 \log_{10} (x)$$

Slope = -20 dB/decade



$$\frac{1}{\tau} = \frac{1}{RC} = 6666.7 \frac{\text{rad}}{\text{sec}}$$

$$= 1061 \text{ Hz} = \text{Bandwidth}$$

## MatLab Commands

```
K = 1; tau = 0.15E-3;
RC_System = tf(K,[tau 1])
bode(RC_System)
```

Semi-log plots: Magnitude (dB) vs.  $\log_{10} f$

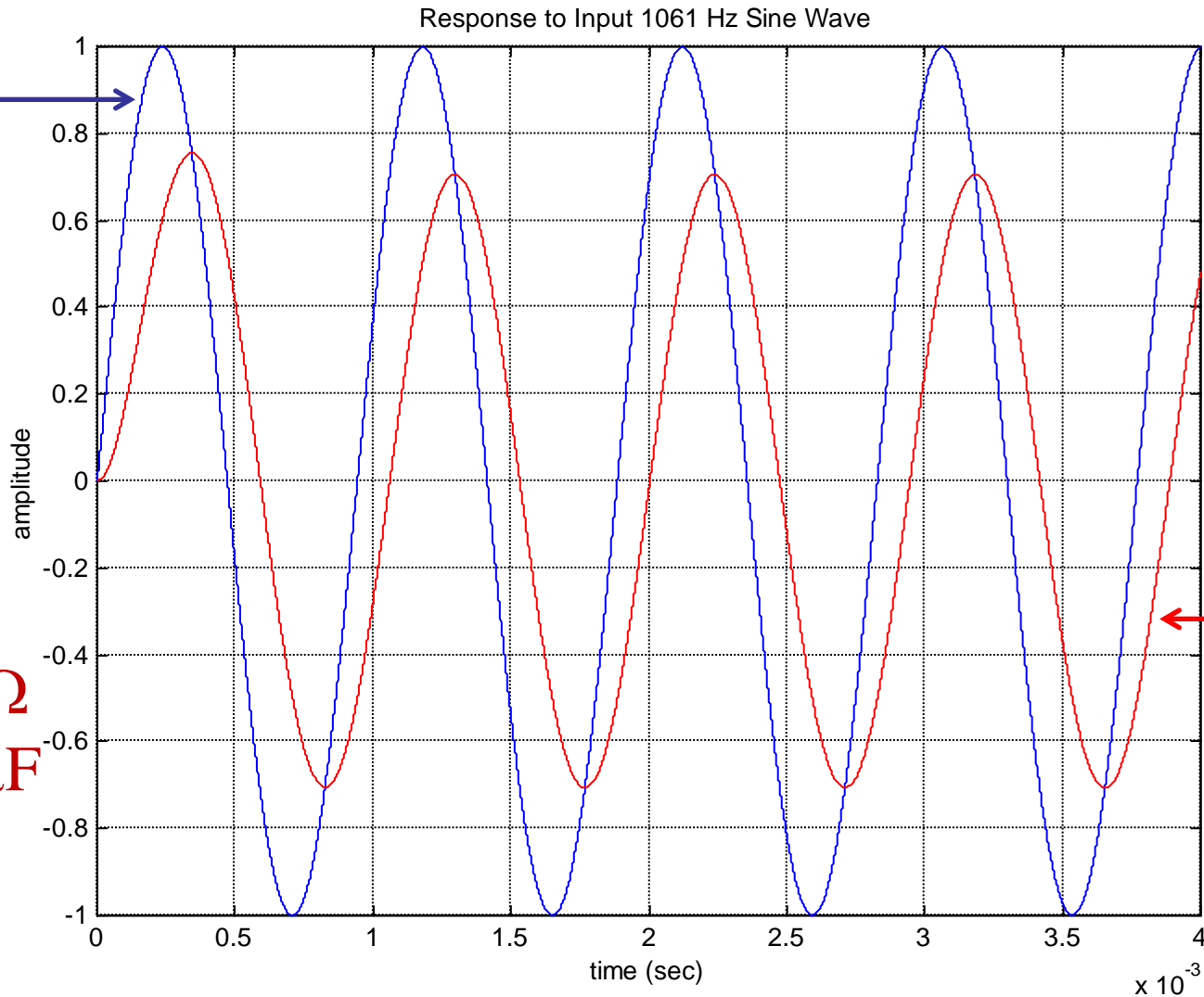
Phase (deg) vs.  $\log_{10} f$

# RC Circuit

Amplitude Ratio =  $0.707 = -3 \text{ dB}$

Phase Angle =  $-45^\circ$

Input  
1061 Hz  
Sine Wave



Output

$R = 15 \text{ K}\Omega$   
 $C = 0.01 \text{ }\mu\text{F}$

- Bandwidth

- The bandwidth is the frequency where the amplitude ratio drops by a factor of  $0.707 = -3\text{dB}$  of its gain at zero or low-frequency.
  - For a 1<sup>st</sup>-order system, the bandwidth is equal to  $1/\tau$ .
  - The larger (smaller) the bandwidth, the faster (slower) the step response.
  - Bandwidth is a direct measure of system susceptibility to noise, as well as an indicator of the system speed of response.
- Note that the amplitude ratio follows low- and high-frequency asymptotes, i.e., slope 0 and  $-20\text{ dB/decade}$ , respectively, and the phase angle approaches  $-90^\circ$  asymptotically. At the break frequency  $1/\tau$ , the phase angle is  $-45^\circ$ .

# Measurements

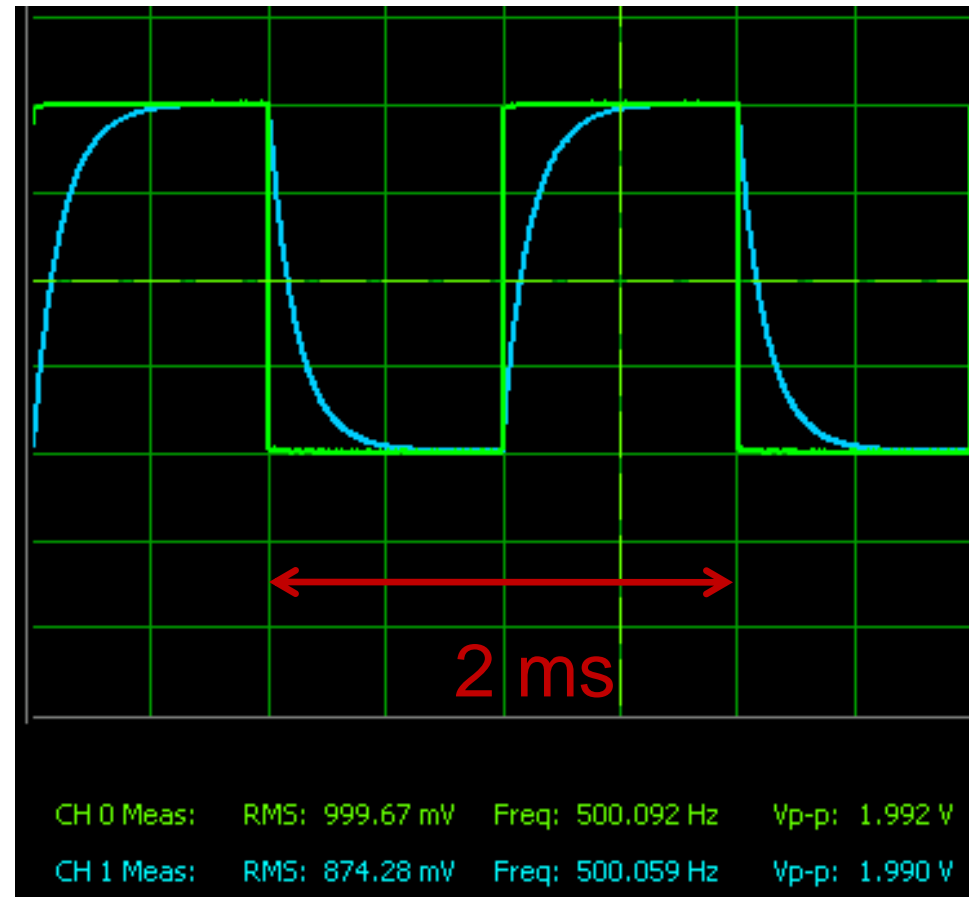
- RC Circuit
  - Time Response (Step Input)
  - Frequency Response (Sine Input)

# Why 500 Hz ?

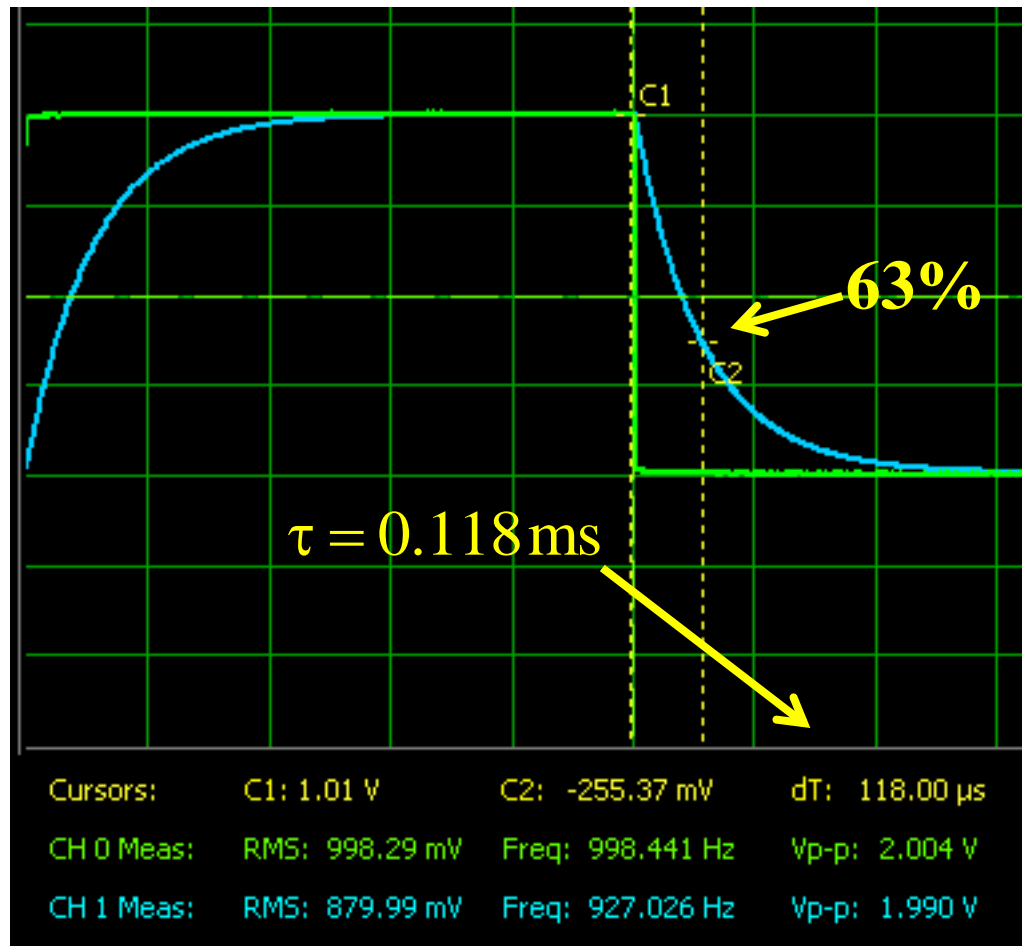
$$5\tau = (5)(0.15\text{ms}) = 0.75\text{ms}$$

Let's Use 1.00 ms:  
Square Wave Period  
2.00 ms

Square Wave Frequency  
500 Hz

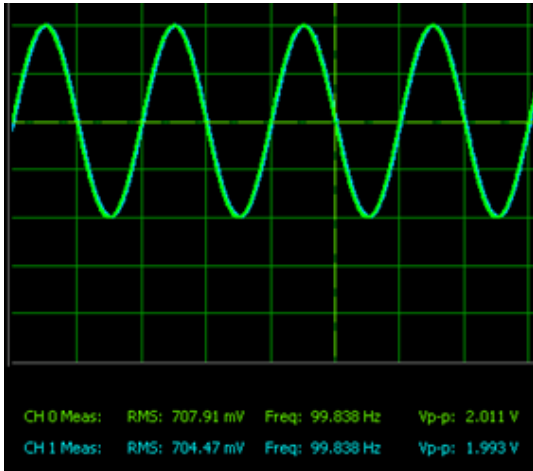


## Step Response: Square Wave Input

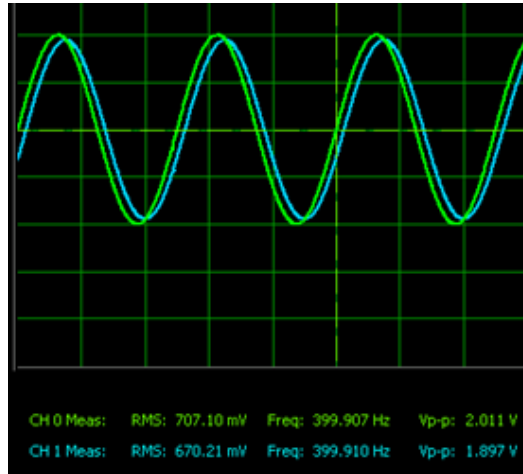


# Frequency Response

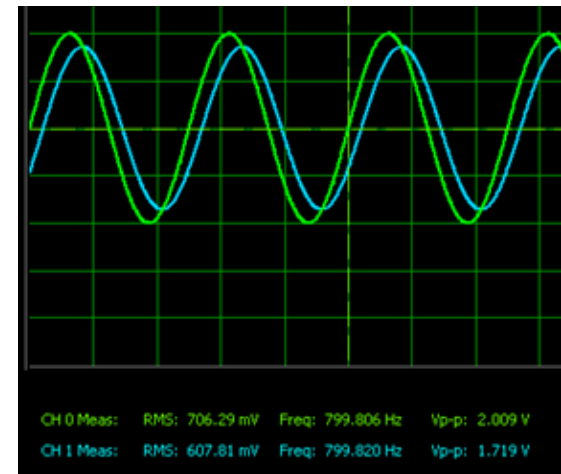
100 Hz



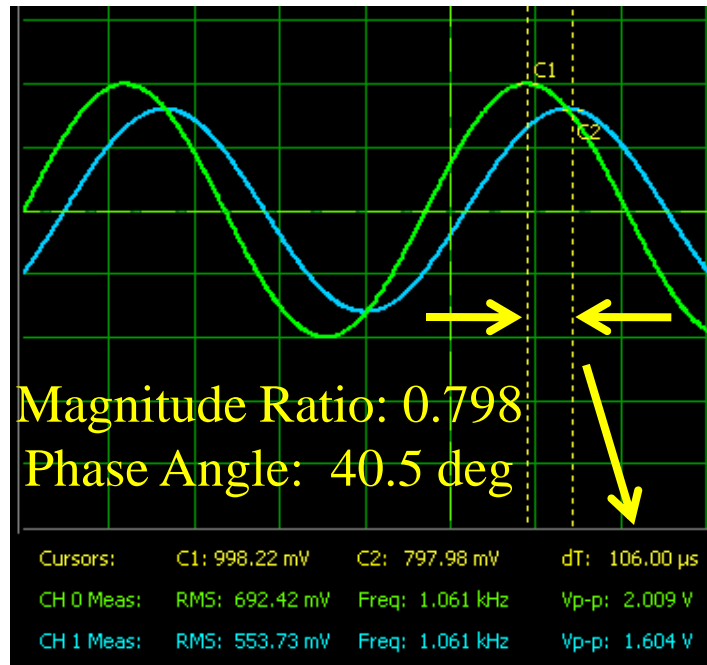
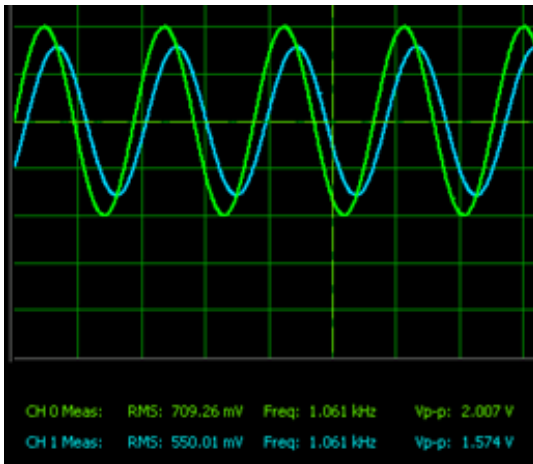
400 Hz



800 Hz

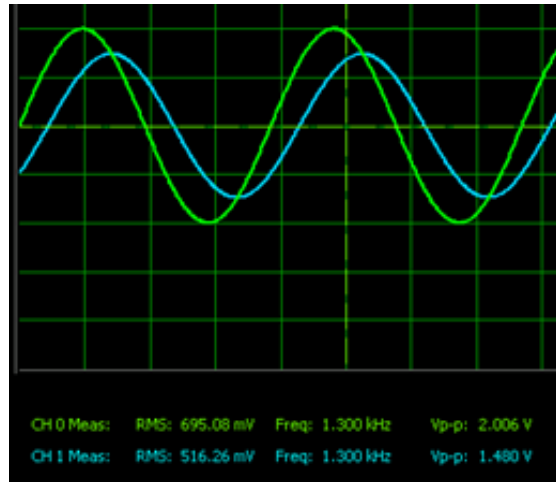


1061 Hz

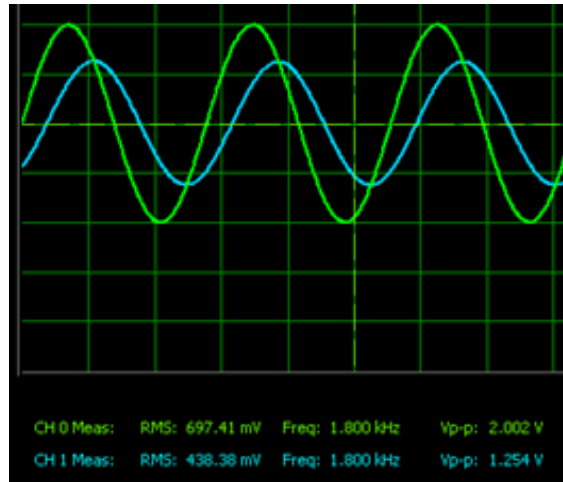


$f = 1061 \text{ Hz}$   
 $T = 0.943 \text{ ms}$

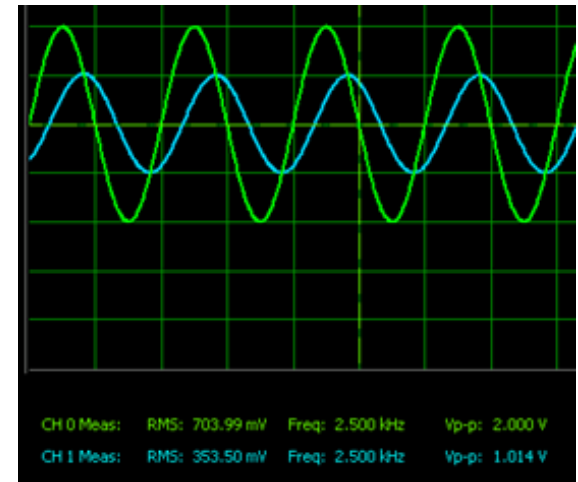
# 1300 Hz



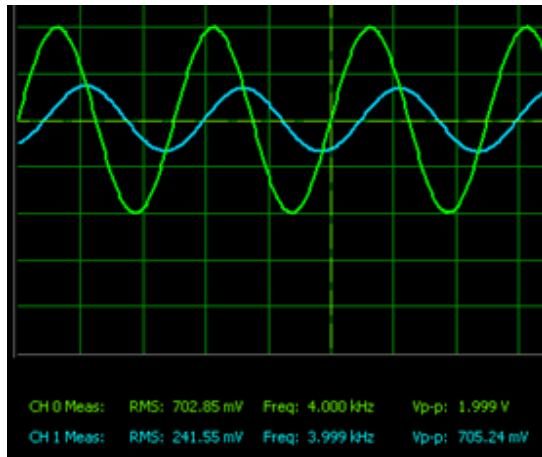
# 1800 Hz



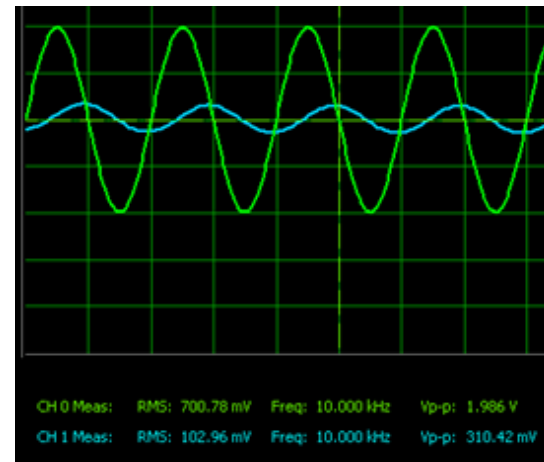
# 2500 Hz



# 4000 Hz



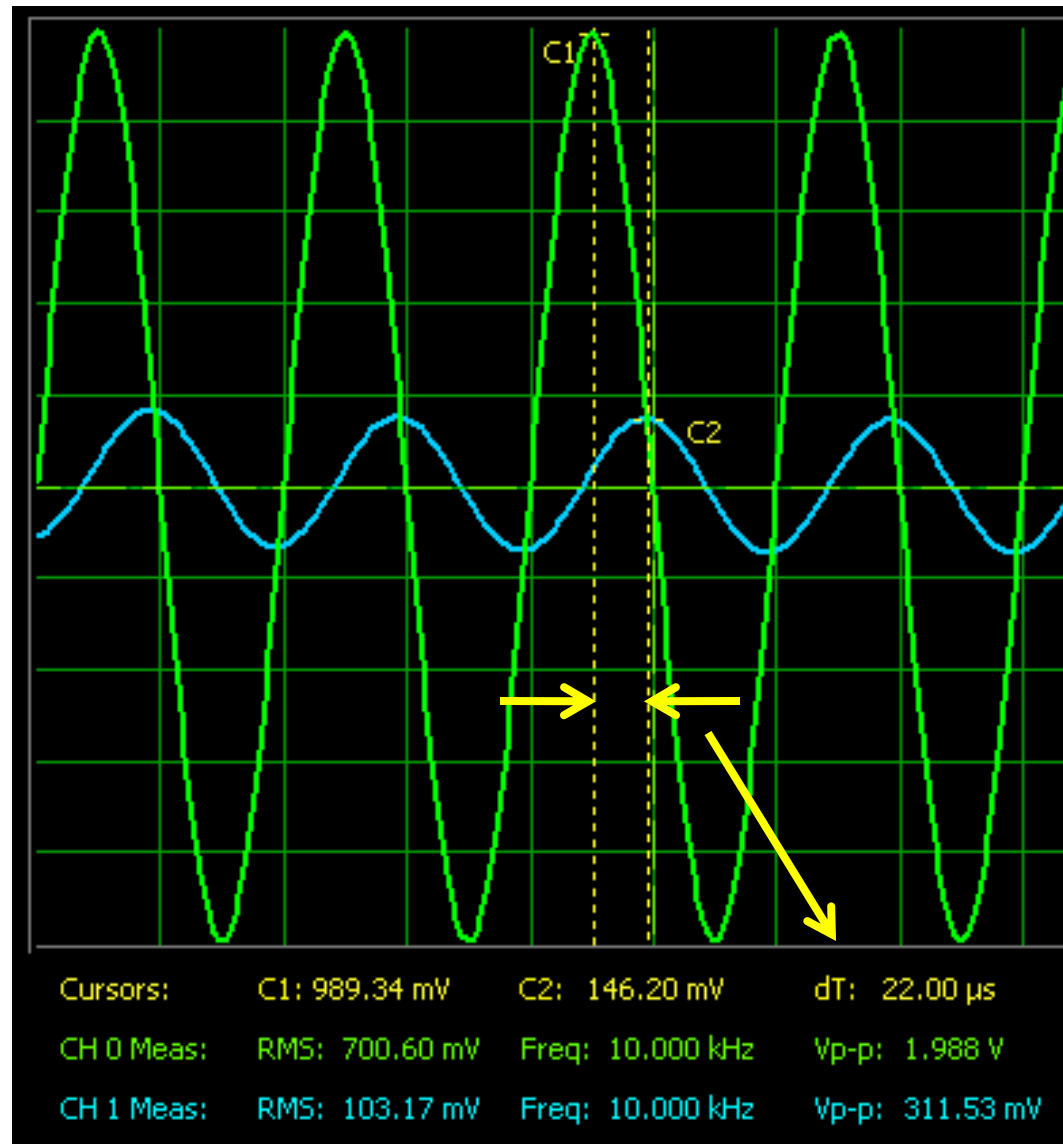
# 10000 Hz





$f = 10 \text{ KHz}$   
 $T = 0.100 \text{ ms}$

Magnitude Ratio:  
0.156  
Phase Angle:  
79.2 deg



## Semi-Log Plots

Amplitude  
Ratio (dB)

## Bode Plot

Phase  
Angle  
(degrees)

