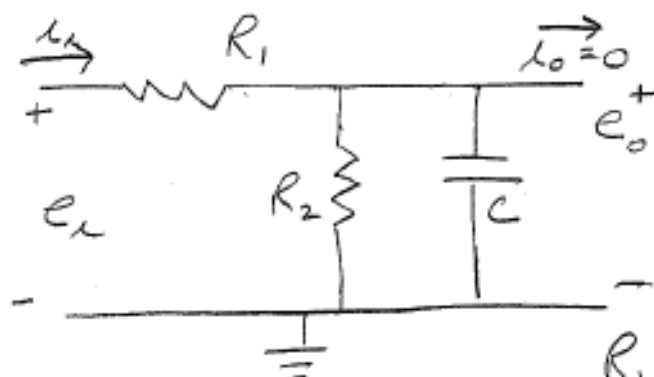


PI Control of a 1st-Order Plant

K. Craig
2/2011

1

Plant \Rightarrow



Assume
No
Loading

$$R_1 = R_2 = 100 \text{ k}\Omega$$
$$C = 1 \mu\text{F}$$

$$\frac{e_o}{e_u} = \frac{\frac{(R_2)(1/CD)}{R_2 + 1/CD}}{R_1 + \frac{(R_2)(1/CD)}{R_2 + 1/CD}}$$
$$= \frac{R_2}{R_1 R_2 C D + R_1 + R_2} = \frac{\frac{R_2}{R_1 + R_2}}{\frac{R_1 R_2}{R_1 + R_2} C D + 1} = \frac{K}{\tau_p D + 1}$$

$$K = \frac{R_2}{R_1 + R_2} = 0.5$$

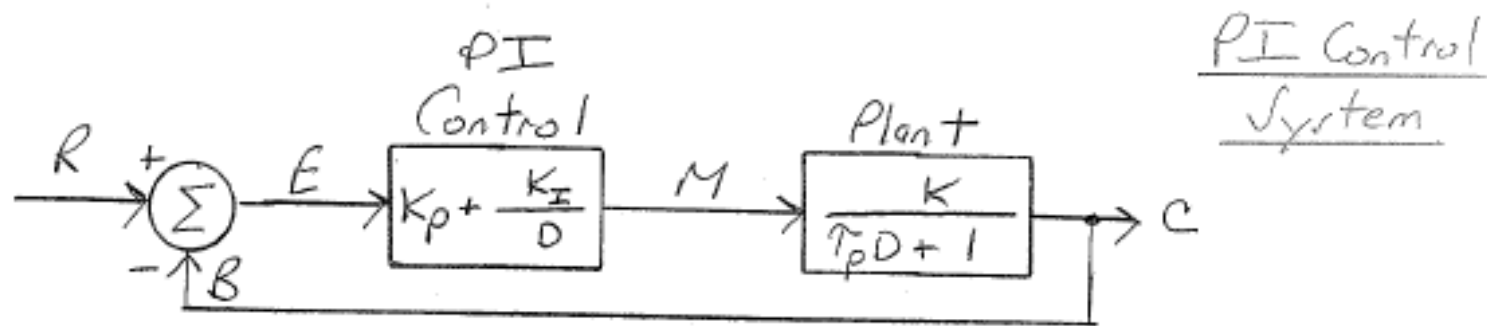
$$\tau_p = \frac{R_1 R_2}{R_1 + R_2} C = 0.05 \text{ sec}$$
$$= 50 \text{ ms}$$

$$\text{Bandwidth} = 1/\tau = 20 \text{ rad/s} = 3.18 \text{ Hz}$$

Performance Specifications:

Unit Step Response

$$\left\{ \begin{array}{l} t_s \leq 0.09 \text{ sec} \\ \quad 1\% \\ t_r \leq 0.015 \text{ sec} \\ \quad 10-90\% \\ M_p \leq 25\% \end{array} \right.$$



$$G = \frac{K}{\tau_p D + 1}$$

$$G_c = K_p + \frac{K_I}{D}$$

$$\begin{aligned} \frac{C}{R} &= \frac{G_c G}{1 + G_c G} = \frac{\left(\frac{K_p D + K_I}{D}\right) \left(\frac{K}{\tau_p D + 1}\right)}{1 + \left(\frac{K_p D + K_I}{D}\right) \left(\frac{K}{\tau_p D + 1}\right)} \\ &= \frac{K_I K \left(\frac{K_p}{K_I} D + 1\right)}{\tau_p D^2 + (K K_p + 1) D + K K_I} \end{aligned}$$

$$\begin{aligned} \frac{C}{R} &= \frac{\frac{K_I K}{\tau_p} \left(\frac{K_p}{K_I} D + 1\right)}{D^2 + \left(\frac{K K_p + 1}{\tau_p}\right) D + \frac{K K_I}{K}} = \frac{\omega_n^2 (\tau D + 1)}{D^2 + 2 \zeta \omega_n D + \omega_n^2} \end{aligned}$$

$$\tau = K_p / K_I$$

$$\omega_n^2 = \frac{K K_I}{\tau_p} \Rightarrow K_I = \frac{\tau_p \omega_n^2}{K}$$

$$2 \zeta \omega_n = \frac{K K_p + 1}{\tau_p} \Rightarrow \zeta = \frac{1 + K_p K}{2 \sqrt{\tau_p K_I K}} \Rightarrow K_p = \frac{1}{K} \left[2 \zeta \sqrt{\tau_p K_I K} - 1 \right]$$

Analog Implementation

(Use Buffer Op-Amps to prevent loading)

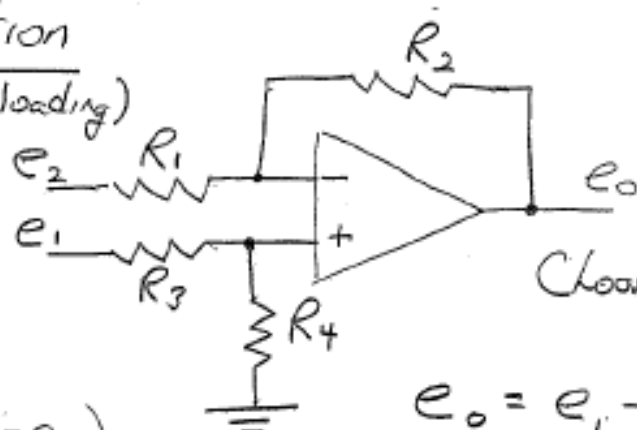
Summing Junction

(Difference Amp)

$$R_1 = R_3$$

$$R_2 = R_4$$

$$e_o = \frac{R_2}{R_1} (e_1 - e_2)$$

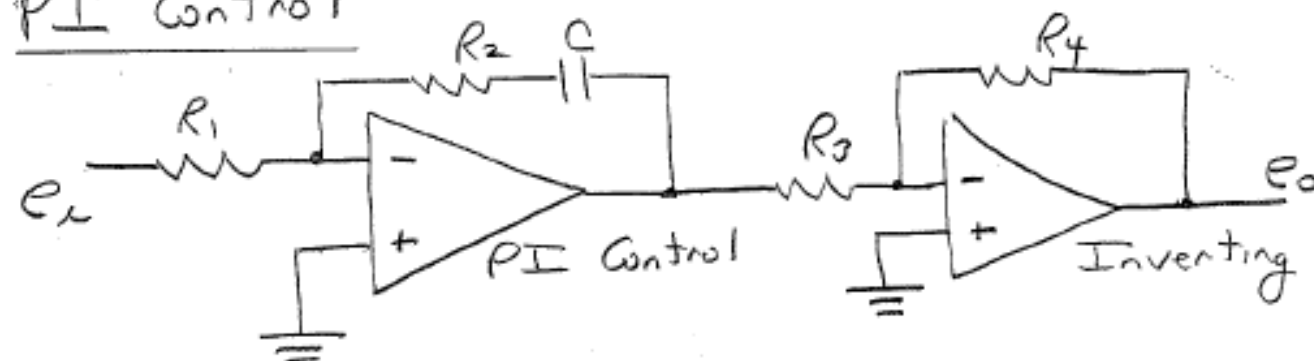


Choose

$$R_1 = R_2 = R_3 = R_4$$

$$e_o = e_1 - e_2$$

PI Control



Avoid
* Saturation
 $e_o < 14V$

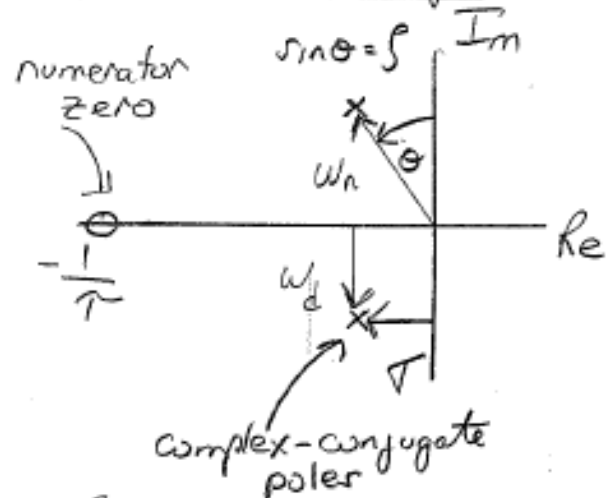
$$\frac{e_o}{e_i} = \left(-\frac{R_4}{R_3} \right) \left(-\frac{R_2 + 1/sC}{R_1} \right) = \left(\frac{R_4}{R_3} \right) \left(\frac{R_2 sC + 1}{R_1 sC} \right)$$

$$K_I = \frac{R_4}{R_3 R_1 C}$$

$$K_P/K_I = R_2 C \Rightarrow K_P = \frac{R_2 R_4}{R_3 R_1}$$

$$\Leftarrow \begin{cases} = \left(\frac{R_4}{R_3 R_1 C} \right) \left(\frac{R_2 sC + 1}{s} \right) \\ = \frac{K_P s + K_I}{s} = \frac{K_I (K_P/K_I s + 1)}{s} \end{cases}$$

Control Design



Pure Second-Order System
w/ complex-conjugate poles.

$$\frac{C}{R} = \frac{\omega_n^2 (\tau D + 1)}{D^2 + 2\zeta\omega_n D + \omega_n^2}$$

poles $\begin{cases} D = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \\ D = -\sigma \pm j\omega_d \end{cases}$

For a pure 2nd-Order System

$$t_r \approx \frac{1.8}{\omega_n}$$

10-90%

$$t_s \approx \frac{4.6}{\zeta\omega_n}$$

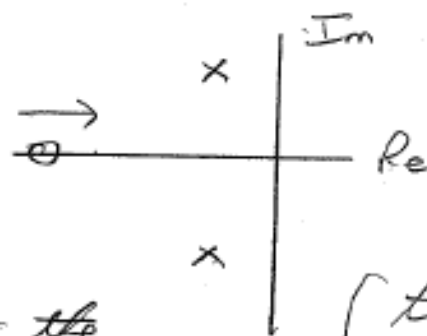
$\pm 1\%$

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

$$t_p = \pi/\omega_d$$

Design Controller to meet
performance specs w/o saturation

However



As the
zero moves
closer to poles

$$\begin{cases} t_r \uparrow \\ t_r \downarrow \\ M_p \uparrow \end{cases}$$

Steps in Process

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- 1) $\left. \begin{matrix} t_r \\ t_s \\ M_p \end{matrix} \right\} \rightarrow \left\{ \begin{matrix} \zeta \\ \omega_n \end{matrix} \right.$
 $\zeta = 0.64 \rightarrow M_p = 7.3\%$
 $\omega_n = 118 \rightarrow t_r = 0.0155$
 $t_s = 0.0615$
- 2) $\left\{ \begin{matrix} \zeta \\ \omega_n \end{matrix} \right\} \rightarrow \left\{ \begin{matrix} K_p \\ K_I \end{matrix} \right.$
- 3) Simulation (Simulink) to evaluate response and control effort.
Note: effect of numerator zero and monitor control effort
- 4) Choose final values for K_p and K_I - verify with simulation.
- 5) $\left\{ \begin{matrix} K_p \\ K_I \end{matrix} \right\}$ Choose R's and C's
- 6) - Build System + Analog PI Controller
 - Measure response
 - Compare to model prediction

$$K_p = 13.1$$

$$K_i = 1392$$

$$\zeta = 0.64$$

$$\omega_n = 118$$

2nd-order
term

$$V = 75 = j\omega_n$$

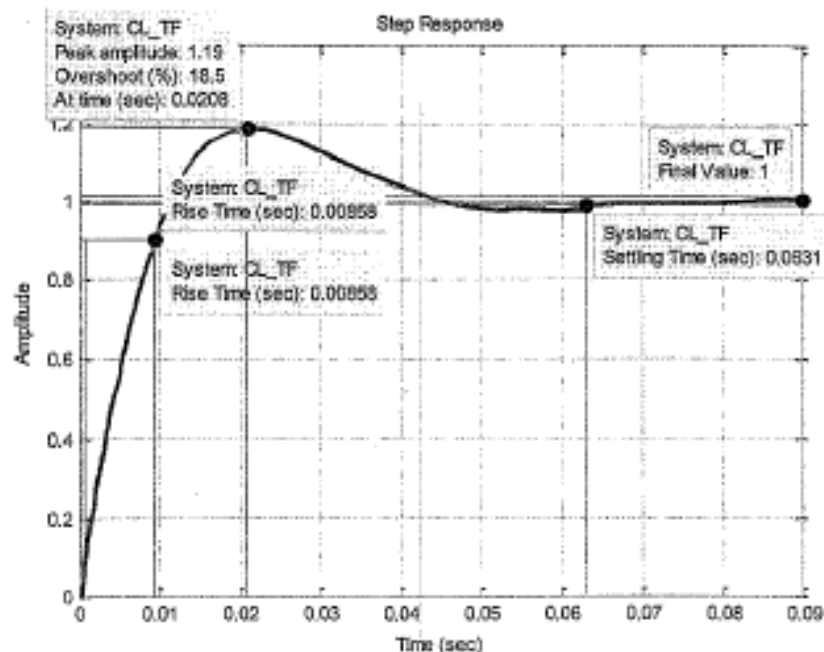
$$\frac{K_i}{K_p} = \frac{1}{T}$$

$$= 106$$

Numerator
Zero has
a large
effect.

$t_r \downarrow$

$\eta_p \uparrow$ $t_s \uparrow$



2nd-Order
System
w/o zero:

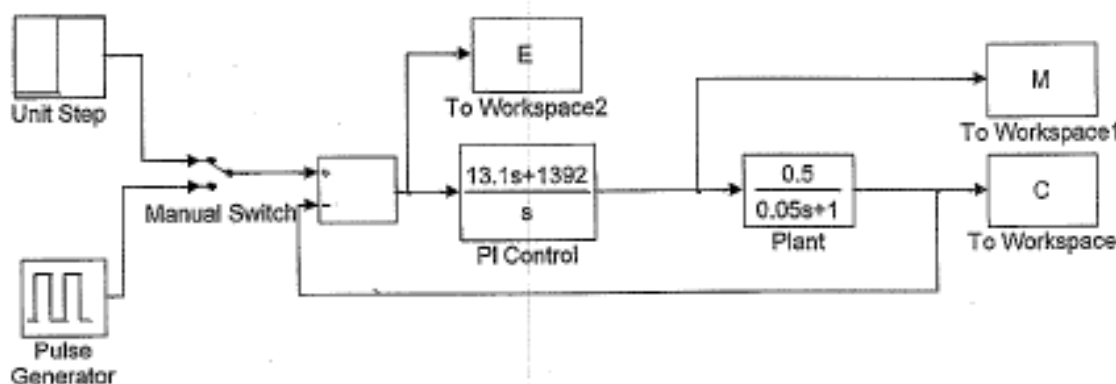
$$\zeta = 0.64 \quad \omega_n = 118$$

$$10-90\% t_r = 0.0166$$

$$1\% t_s = 0.0543$$

$$\eta_p = 7.3\%$$

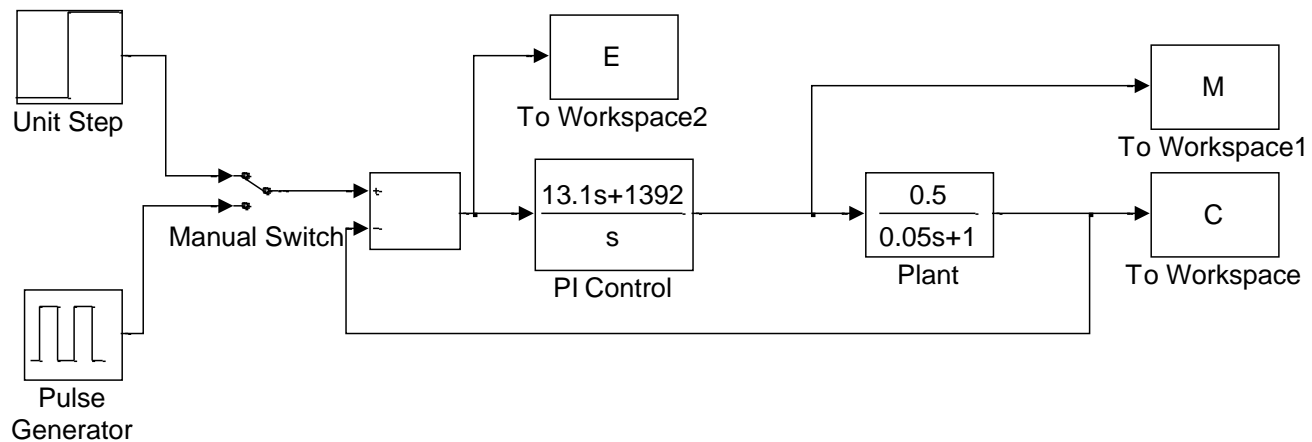
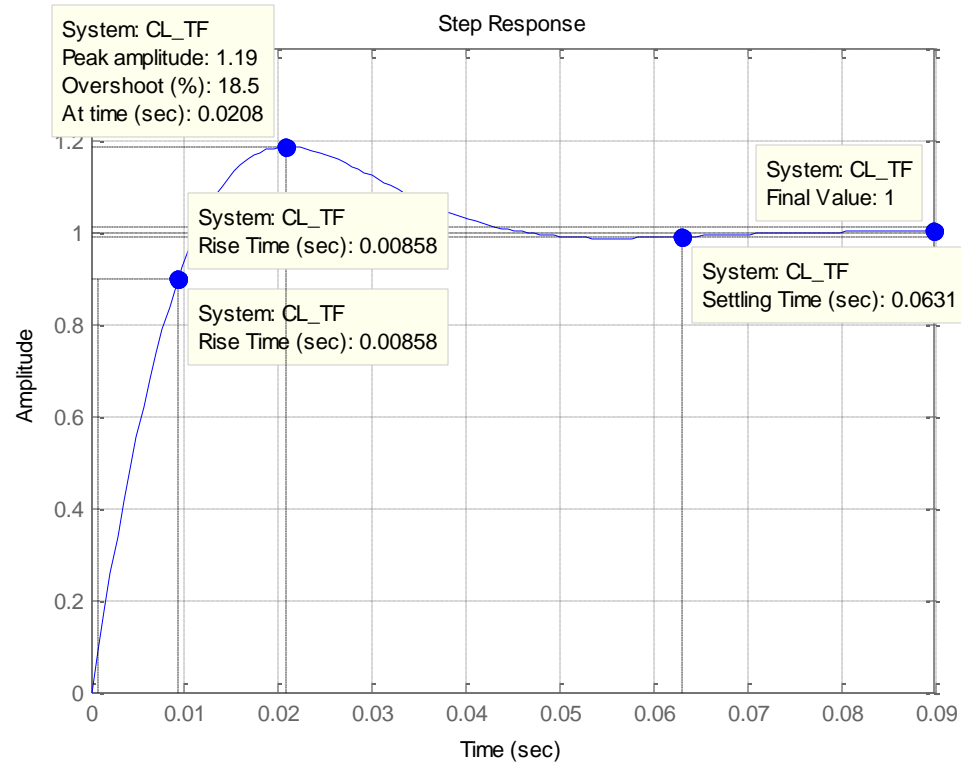
$$t_p = 0.0347 \text{ sec}$$



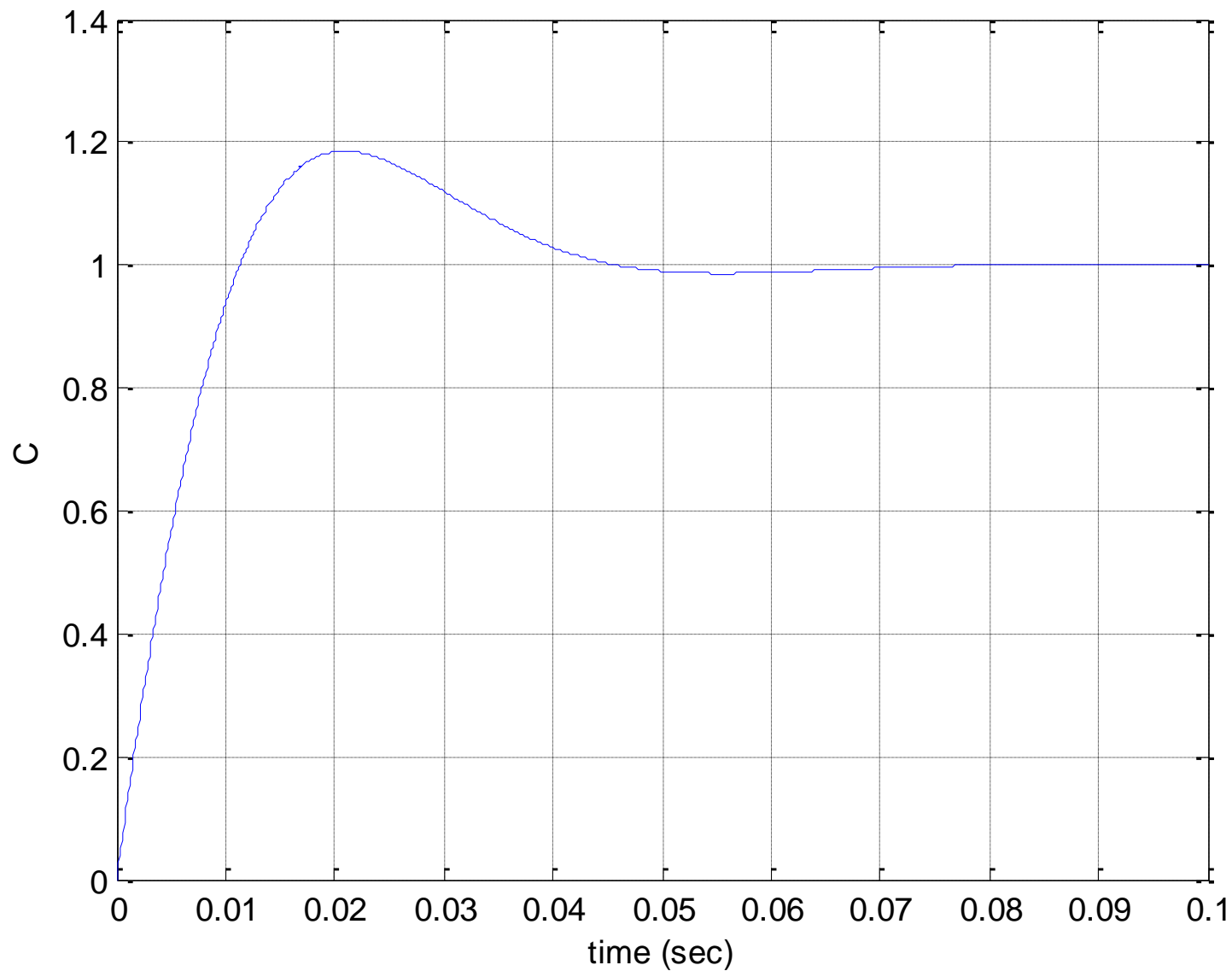
$M < 14V$

$$K_p = 13.1$$

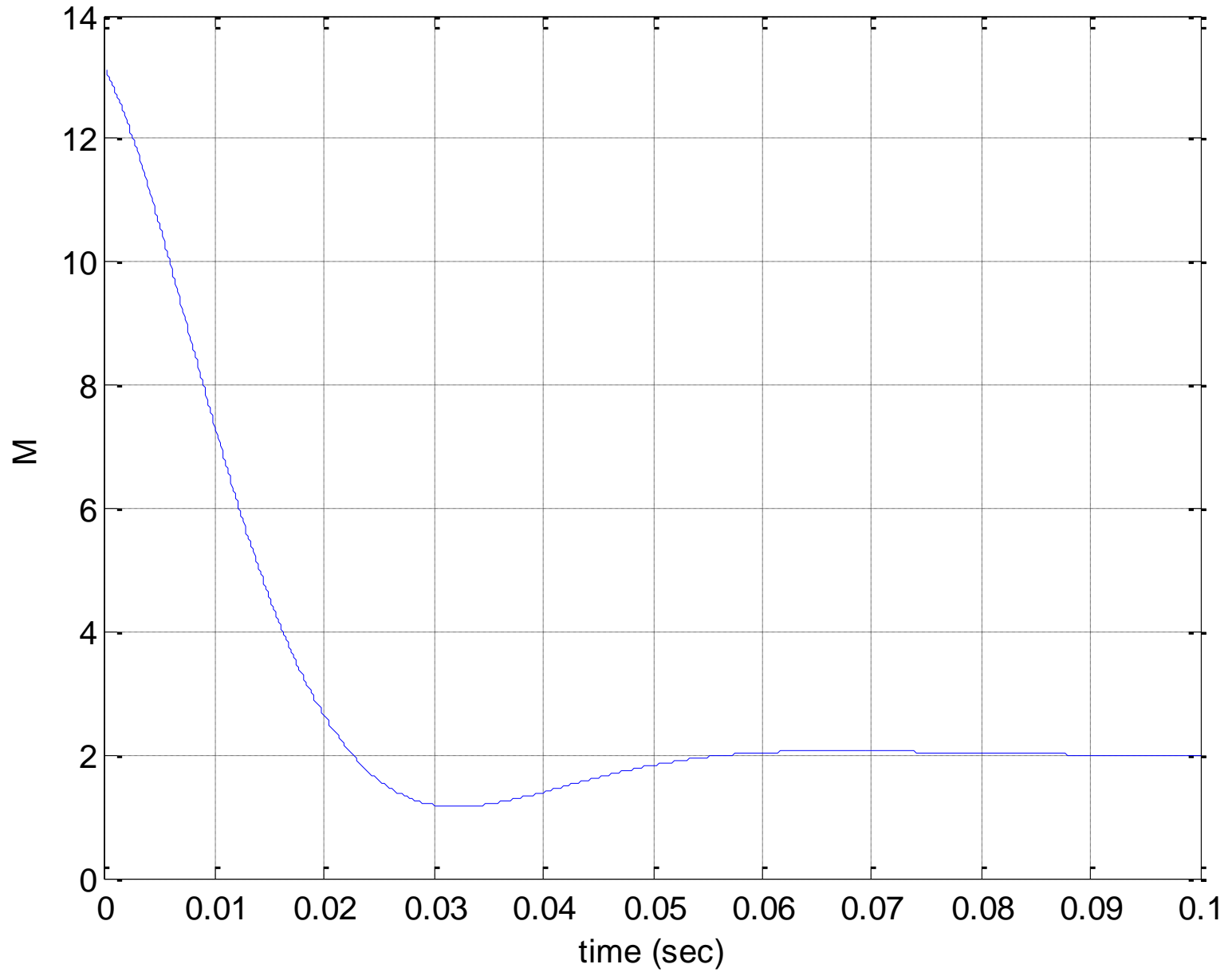
$$K_i = 1392$$



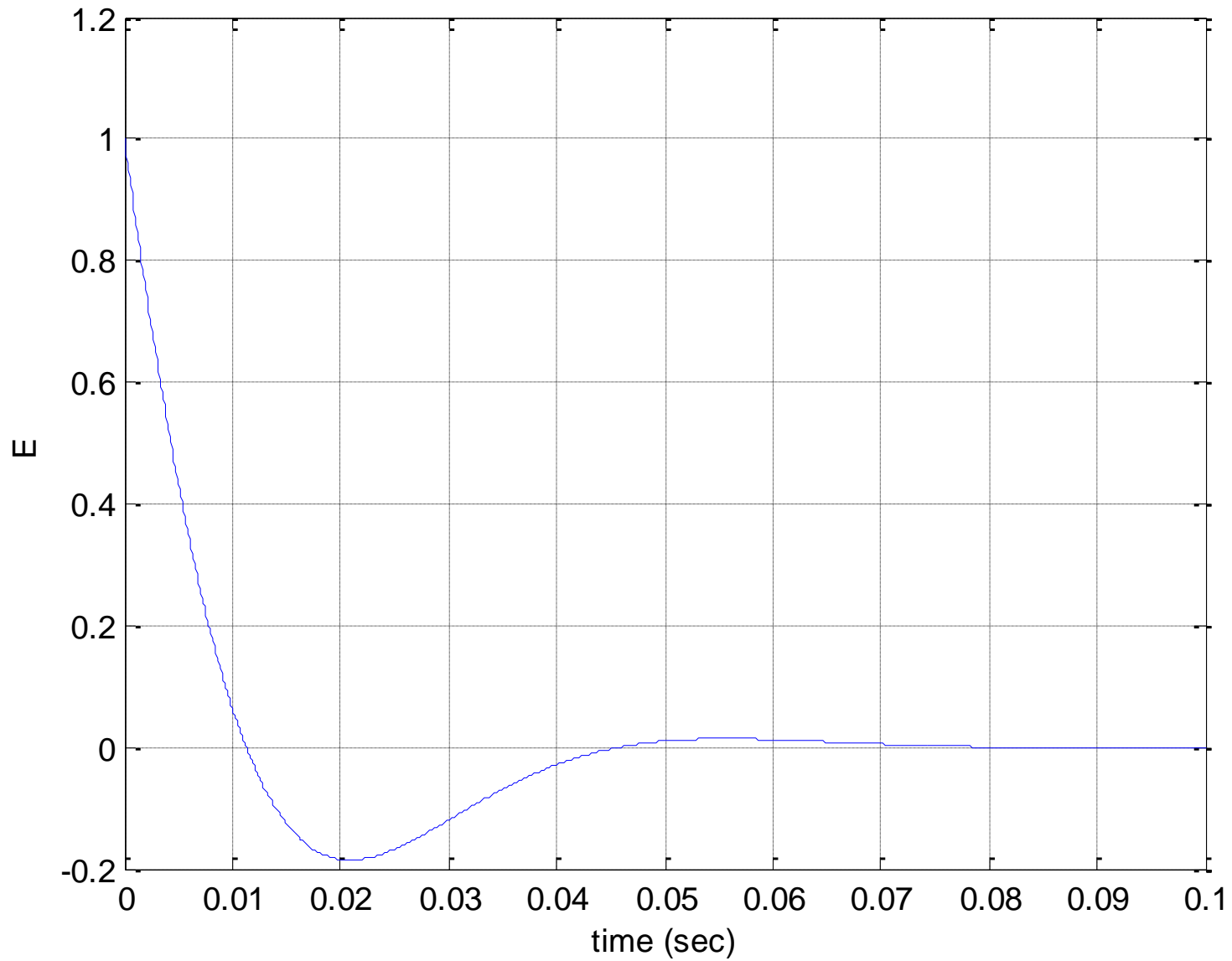
C vs. time

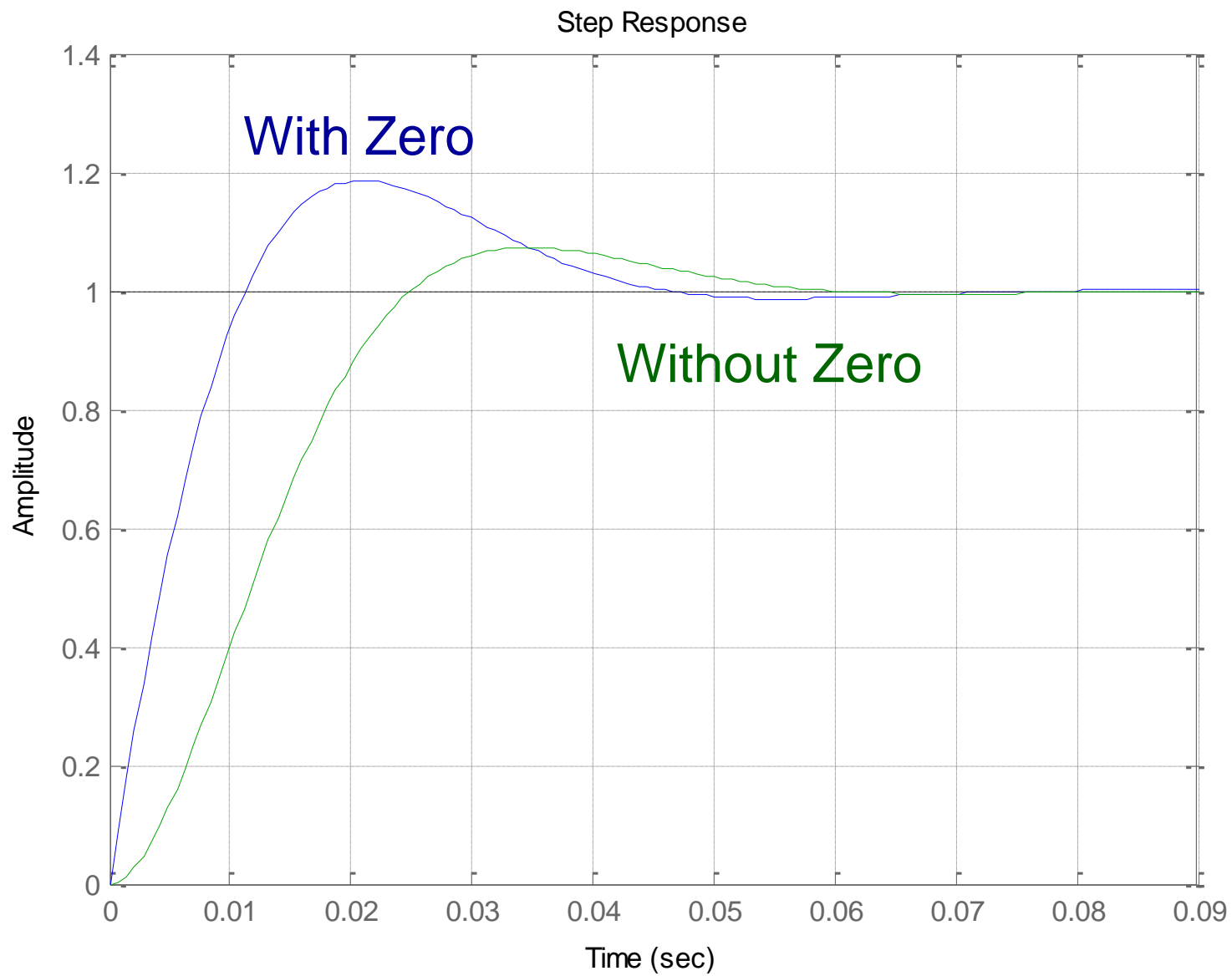


M vs. time

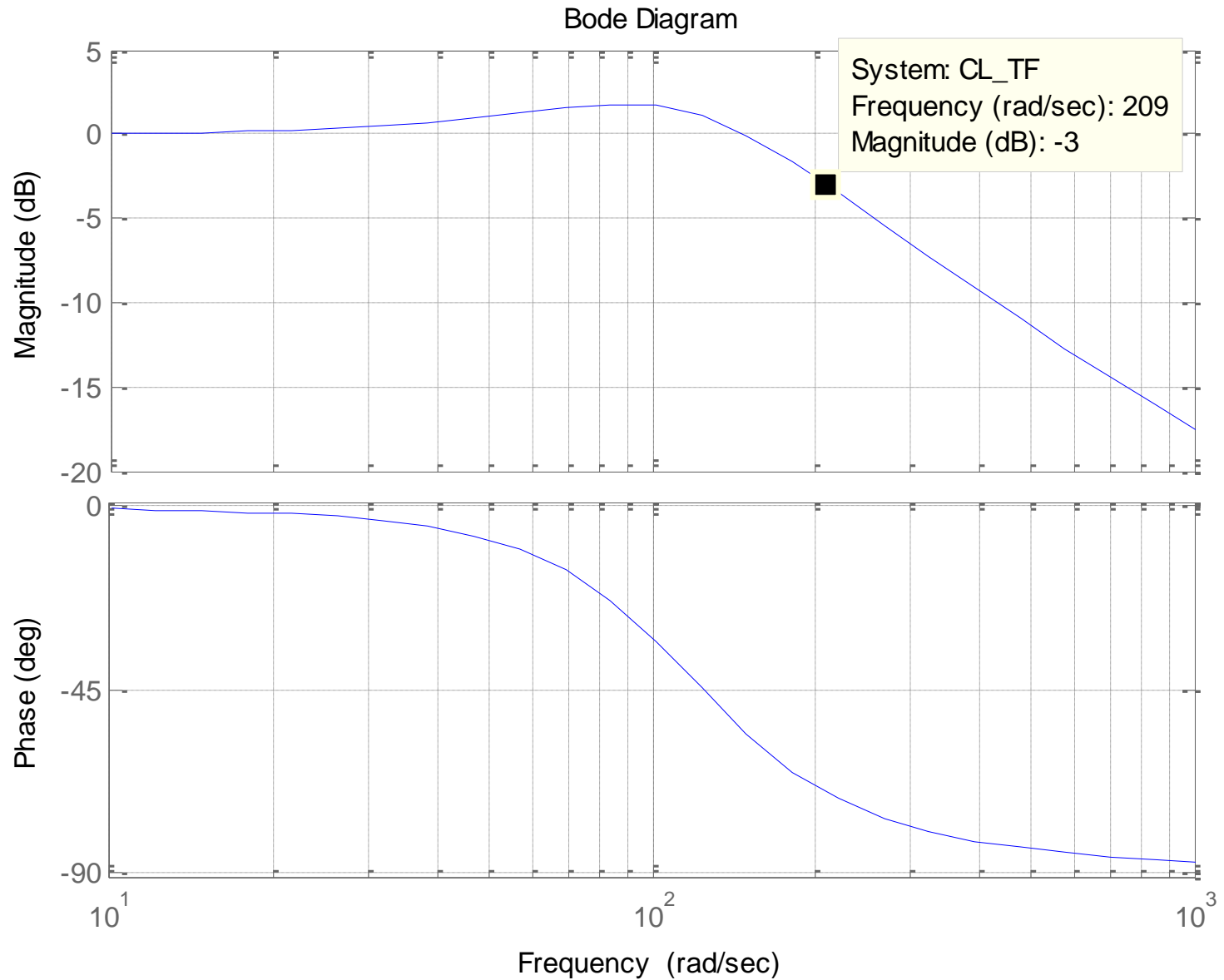


E vs. time





Closed-Loop Analytical Bode Plot



Actual System

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$$\left. \begin{aligned} K_I &= \frac{R_4}{R_3 R_1 C} = 1392 \\ K_P &= \frac{R_2 R_4}{R_3 R_1} = 13.1 \end{aligned} \right\} \left. \begin{aligned} \frac{13.1}{R_2 C} &= 1392 \\ \frac{R_4}{R_3 R_1} &= \frac{13.1}{R_2} \end{aligned} \right\} R_2 C = .0094$$

$$\left. \begin{aligned} R_2 &= 9.1 \text{ K}\Omega \\ C &= 1 \mu\text{F} \\ R_2 C &= .0091 \end{aligned} \right\}$$

$$\frac{R_4}{R_3 R_1} = \frac{13.1}{9.1 \text{ K}\Omega} = .0014$$

$$\frac{R_4}{R_3} = (R_1)(.0014)$$

$$R_1 = 1 \text{ K}\Omega$$

$$\frac{R_4}{R_3} = 1.4$$

$$R_4 = 1.8 \text{ K}\Omega$$

$$R_3 = 1.3 \text{ K}\Omega$$

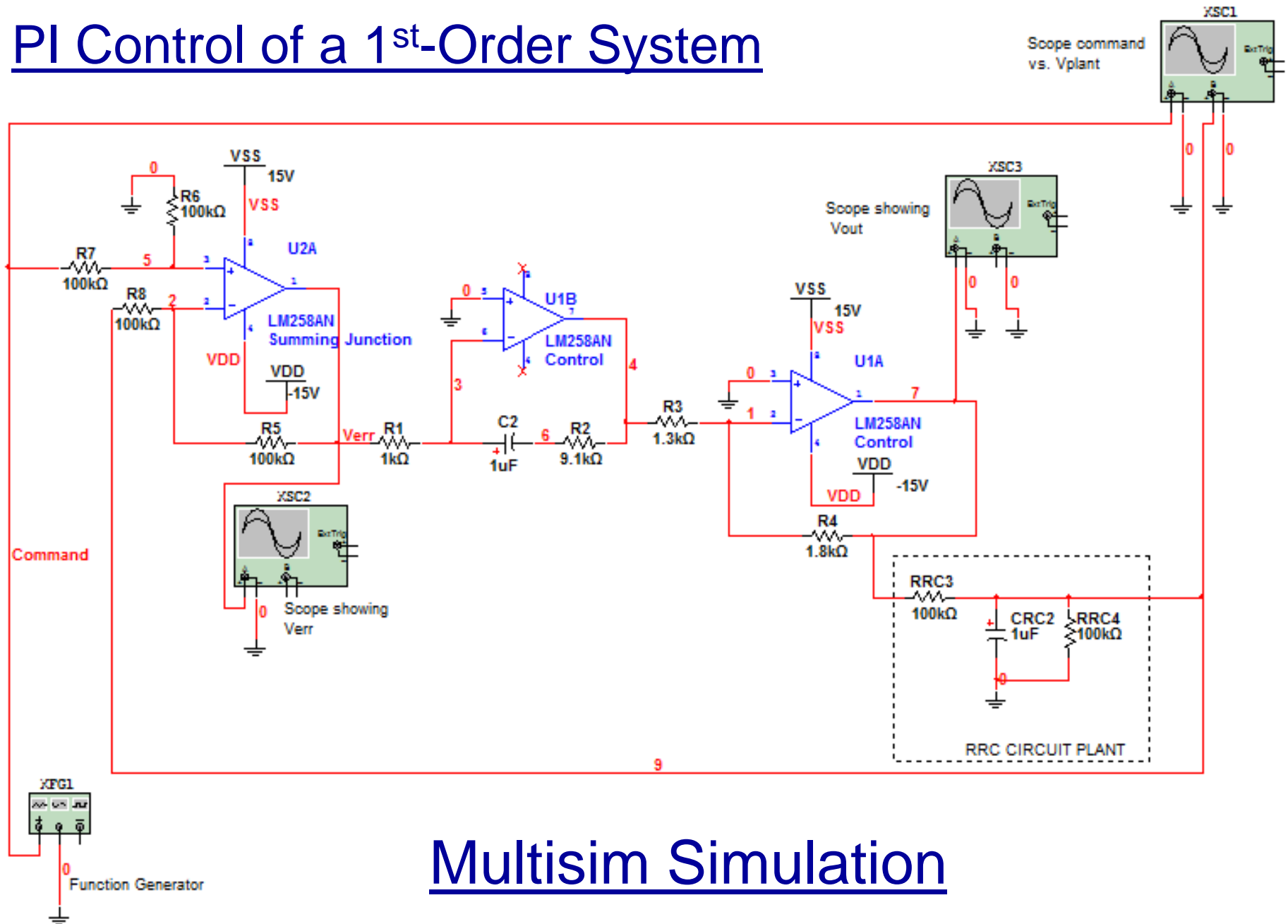
Summary:

$$R_1 = 1 \text{ K}\Omega \quad R_3 = 1.3 \text{ K}\Omega$$

$$R_2 = 9.1 \text{ K}\Omega \quad R_4 = 1.8 \text{ K}\Omega$$

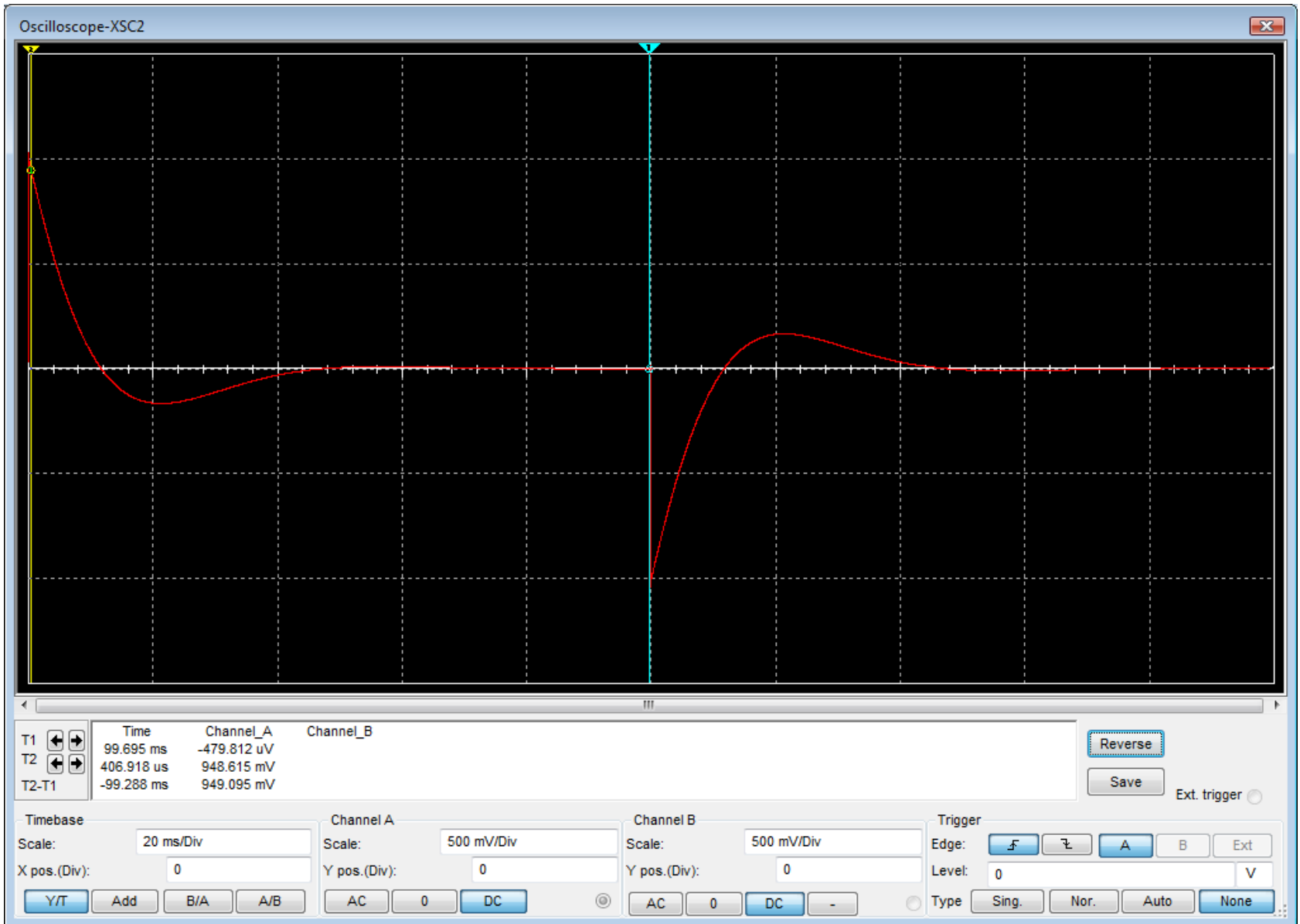
$$C = 1 \mu\text{F}$$

PI Control of a 1st-Order System

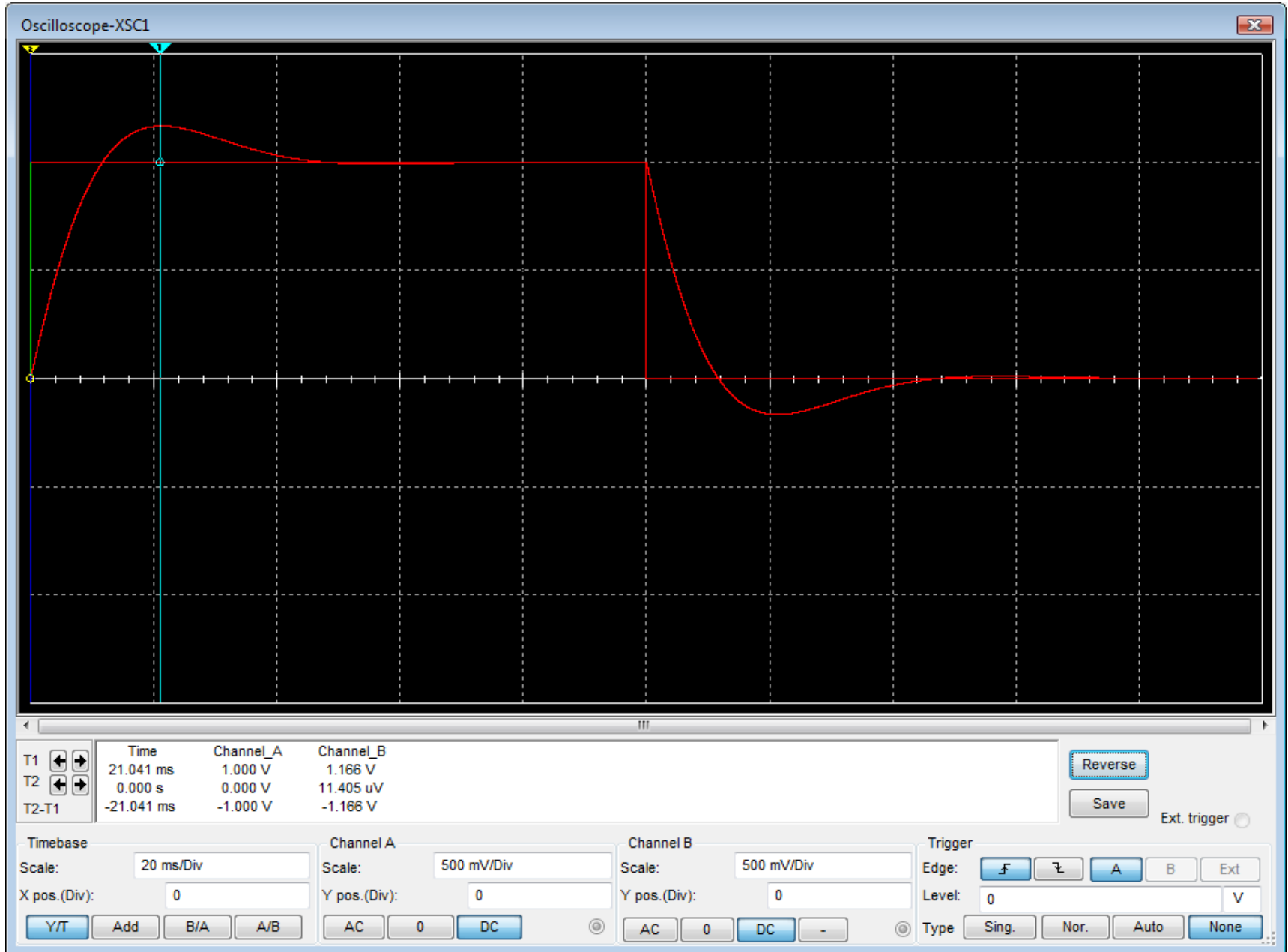


Multisim Simulation

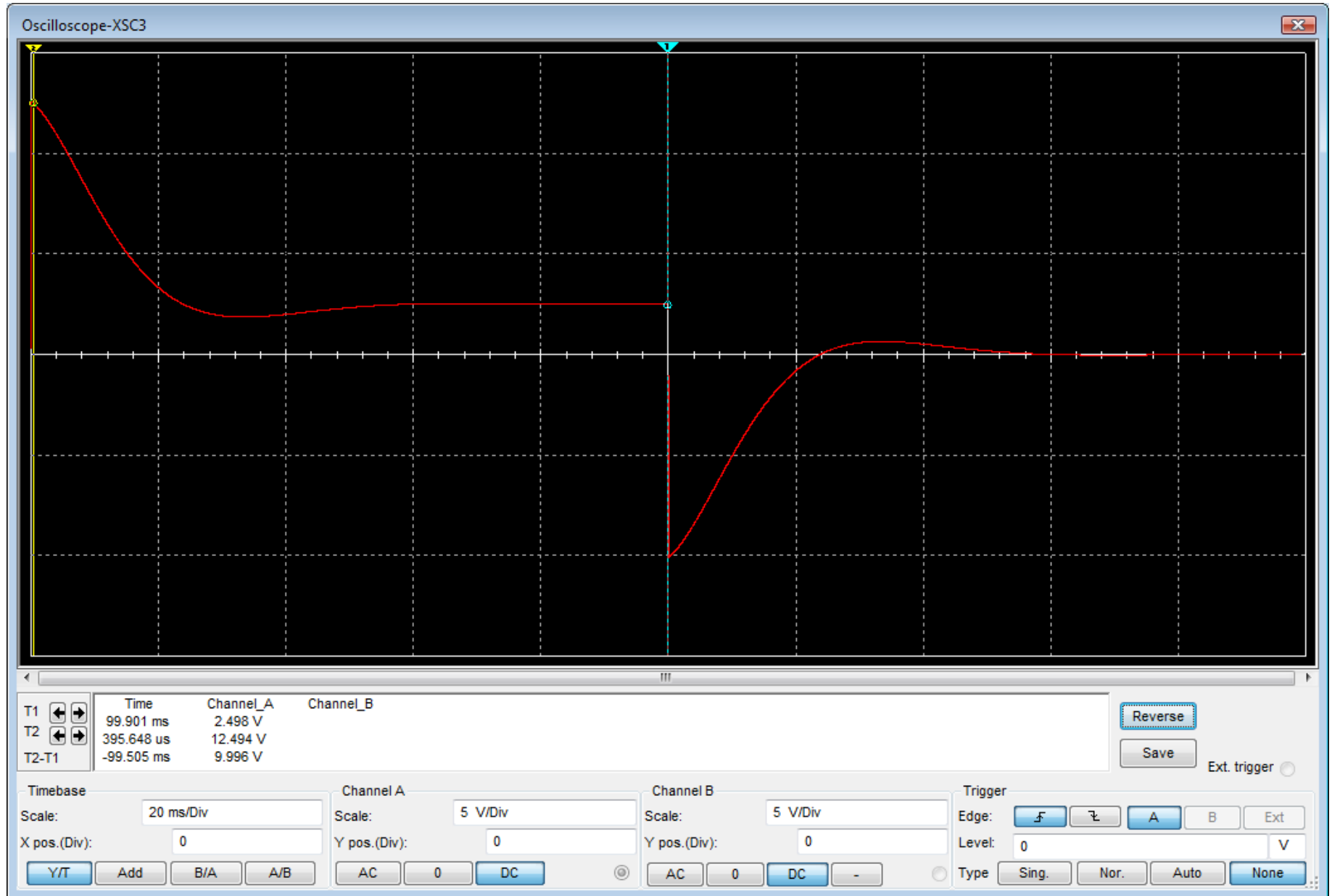
Error Signal



System Response

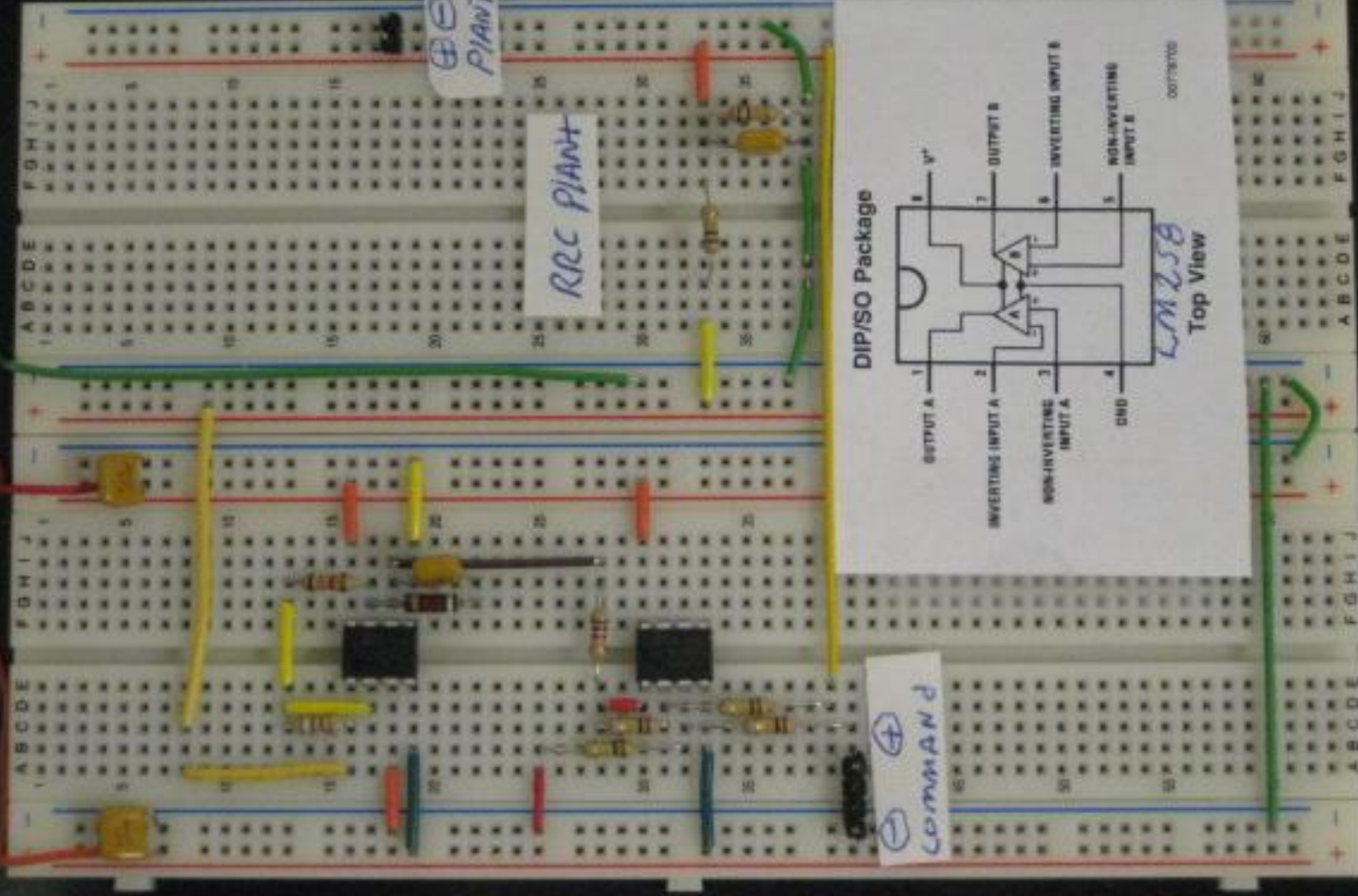
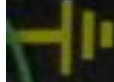


Control Effort



V_a

V_b





Function Generator - NI ELVISmx


LabVIEW

5.0000 Hz

Waveform Settings







Frequency

200m

20k

5 Hz

Amplitude

0.0

10.0

1.00 Vpp

DC Offset

-5.0

5.0

0.50 V

Duty Cycle

50 %

Modulation Type

None

Sweep Settings

Start Frequency

100.0 Hz

Stop Frequency

1.0k Hz

Step

100.00 Hz

Step Interval

1000 ms

Instrument Control

Device


Dev2 (NI myDAQ)

Signal Route


AO 0

☐ Manual Mode


Run




Sweep



Stop

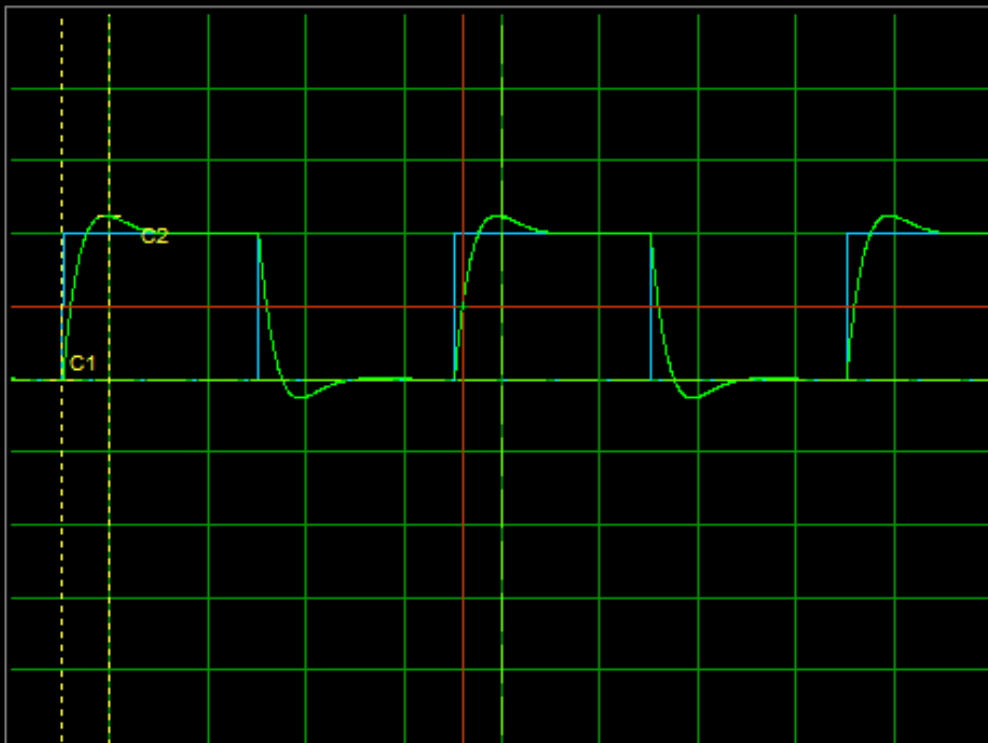


Help





Sample Rate: 5.00 kS/s



Cursors: C1: -1.82 mV C2: 1.12 V dT: 24.20 ms Timeout

CH 0 Meas: RMS: 743.53 mV Freq: 4.998 Hz Vp-p: 1.001 V

CH 1 Meas: RMS: 739.05 mV Freq: 4.999 Hz Vp-p: 999.96 mV

Cursors Settings

☒ Cursors On C1 CH 0 C2 CH 0

Display Measurements

☒ CH 0 ☒ CH 1

Basic Settings

Advanced Settings

Channel 0 Settings

Source

AI 0

☒ Enabled

Probe

1x

Coupling

DC

Scale
Volts/Div

500 mV

Vertical
Position (Div)

0

Channel 1 Settings

Source

AI 1

☒ Enabled

Probe

1x

Coupling

DC

Scale
Volts/Div

500 mV

Vertical
Position (Div)

0

Timebase

Time/Div



50 ms

Trigger

Type

Edge

Slope



Source

Chan 0 Source

Level (V)

0.5

Horizontal Position (%)



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Instrument Control

Device

Dev2 (NI myDAQ)

Acquisition Mode

Run Continuously

Autoscale

Run



Stop



Log



Help



Closed-Loop Experimental Bode Plot

